Важливу роль у вирішенні тривимірних задач теорії пружності грають асимптотичні методи. У рішенні проблеми граничного переходу від тривимірних задач до двовимірних для пружних оболонок особливе місце займає метод асимптотичного інтегрування тривимірних рівнянь теорії пружності. На основі методу асимптотичного інтегрування рівнянь теорії пружності вивчається осісиметрична задача теорії пружності для радіально-неоднорідного циліндра малої товщини. Розглядається випадок, коли модулі пружності змінюються по радіусу за лінійним законом. Передбачається, що бічна частина циліндра вільна від напружень, а на торцях циліндра задані граничні умови, що залишають циліндр в рівновазі. Сформульована крайова задача зводиться до спектральної задачі. Вивчається поведінка рішень спектральної задачі як у внутрішній частині циліндра, так і поблизу торців циліндра при прагненні до нуля параметра тонкостінних циліндра. Отримано три групи рішень і роз'яснено характер побудованих однорідних рішень. Рішення, відповідне першому ітераційному процесу, визначає проникаючий напружено-деформований стан циліндра. Рішення, відповідне другому ітераційного процесу, являє собою крайові ефекти в прикладній теорії оболонок. Третій ітераційний процес визначає рішення, яке має характер пограничного шару. Рішення, відповідне першому і другому ітераційним процесам, визначає внутрішній напружено деформований стан циліндра. У першому члені асимптотики їх можна розглядати як рішення з прикладної теорії оболонок. Показано, що напружено-деформований стан, як і у випадку однорідного циліндра малої товщини, складається з трьох типів: проникаючого напруженого стану, простого крайового ефекту і пограничного шару. Розглянуто питання про задоволення граничних умов на торцях радіально-неоднорідного циліндра з використанням варіаційного принципу Лагранжа

Ключові слова: радіально-неоднорідний циліндр, асимптотичний метод, пограничний шар, крайовий ефект, варіаційний принцип, головний вектор, власне значення

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ANALYSIS OF AXISYMMETRIC PROBLEM FROM THE THEORY OF ELASTICITY FOR AN ISOTROPIC CYLINDER OF SMALL THICKNESS WITH ALTERNATING ELASTICITY MODULES

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1. Introduction

The studies of non-homogeneous shells take one of the special places in the theory of shells. Analysis of non-homogeneous shells based on three-dimensional equations of elasticity theory is a very laborious task. That is why it is necessary to turn to various approximate methods that make it possible to simplify the calculation of shells. The complex nature of the phenomena that occur at deformation of non-homogeneous shells led to the creation of many applied theories, each of which is based on a certain system of assumptions. In modern engineering, there occur the new shelled designs, the calculation of which is impossible within the framework of the existing applied theories. For example, the simplest variants of applied theories do not give a full description of deformation of shells, the thicknesses of which are comparable with longitudinal dimensions and radii of curvature.

To find the region of applicability of existing theories of non-homogeneous shells and to create new and more refined applied theories, it is necessary to analyze the stressed-strained state of inhomogeneous shells from the positions of three-dimensional equations of the elasticity theory. In addition, many issues related to the study of stressed-strained state for non-homogeneous shells can be properly solved only within the framework of the elasticity theory. This is especially important when researching non-stationary and stationary oscillations in a rather wide range of frequencies, when studying stress concentration near the boundary, local loads and openings.

Among the reasons that encourage the study of non-homogeneous shells from the perspective of three-dimensional equations of elasticity theory, most adequately taking into account their mechanical and geometrical structure, it is possible to highlight another one, purely internal. Development of the theory of shells, as well as of any other theory goes from simple to more complex models. That is why the internal logic of development requires the analysis of non-homogeneous shells from more general positions.

2. Literature review and problem statement

Asymptotic methods have made a significant contribution to the development of the theory of plates and shells. These methods have proved to be very effective in studying the problem of limiting transition from three-dimensional to two-dimensional problems of elasticity theory [1–3]. The study of three-dimensional problems of the theory of elasticity for a cylinder was the subject of several studies. Paper [4] reports an asymptotic analysis of the spatial problem of elasticity theory for an isotropic cylinder of small thickness and comparison of asymptotic solution with the solutions obtained by the applied theories. Paper [3] developed a general theory of a transversely-isotropic cylinder of the small thickness, which includes the methods for constructing heterogeneous and homogeneous solutions, which make it possible to reveal the characteristics of stressed-strained state of an anisotropic cylindrical shell. In papers [2, 3], using the method of homogenous solutions, an axisymmetric dynamic problem of the elasticity theory for an isotropic and transversely-isotropic hollow cylinder of small thickness was explored. Homogeneous solutions that depend on the roots of a dispersion equation were constructed. The qualitative study of some applied theories was carried out, the boundaries of their applicability were established. Article [5] studied the axisymmetric problem of the elasticity theory for a radially layered cylinder with alternating hard and soft layers. The theorem, which establishes lamination of the spectrum into the lower and higher part, was obtained. It was shown that there are weakly damped boundary-layer solutions for a radially-layered cylinder with alternating hard and soft layers. They can significantly influence the internal stressed-strained state of a cylinder, which indicates a violation of the Saint-Venant principle in its classical statement. In paper [5], the propagation of axisymmetric waves in a radially-laminated cylinder was studied and the dispersion curves based on a combination of analytical and numerical methods were plotted. Article [6] deals with bending deformation of a multi-layered cylinder with the most common form of cylindrical anisotropy. In paper [7], a semi-analytical method to solve the problem of Almansi-Michell for an inhomogeneous anisotropic cylinder was proposed. Paper [8] considered the equivalent to the Lamé classical problem for an isotropic hollow cylinder with the Jung module, depending on the radial coordinate and with constant Poisson's coefficient. In article [9], the influence of non-homogeneity of the material on the stressed-strained state of a cylinder was explored.

We will note that the analytical solution to a spatial problem of elasticity theory for a radially non-homogeneous cylinder of small thickness is related to significant mathematical difficulties. From the mathematical point of view, studying a spatial problem of the elasticity theory for a radially non-homogeneous cylinder is reduced to the study of boundary-value problems for systems of linear differential equations of the second order in private derivatives with arbitrary variable coefficients. When building homogeneous solutions for radially non-homogeneous cylinder, a spectral problem for differential operators with arbitrary variable coefficients is obtained.

This causes difficulties in the methods for solving problems for radially non-homogeneous cylinders.

That is why it is advisable to consider a classic task of the mathematical theory of elasticity for a radially non-homogeneous cylinder of small thickness using the method of asymptotic integration of equations of the theory of elasticity.

3. The aim and objectives of the study

The aim of this study is to reveal the features and to construct effective methods for calculating the stressed-strained state of a radially inhomogeneous cylinder.

To accomplish the aim, the following tasks have been set:

– to construct homogeneous solutions using the method of asymptotic integration of equations of the elasticity theory, based on three iterative processes;

– to construct asymptotic formulas for displacements and stresses;

– to analyze the stressed-strained states, corresponding to various types of homogenous solutions;

– meeting the boundary conditions at the ends of a cylinder.

4. Statement of boundary-value problems for a radially inhomogeneous cylinder

Consider the axisymmetric problem of the elasticity theory for a radially inhomogeneous isotropic hollow cylinder of small thickness. In the cylindrical coordinate system, the area occupied by the cylinder will be designated as:

$$\Gamma=\left\{r\in[r_1;r_2],\ \phi\in[0,2\pi],\ z\in[-L;L]\right\}.$$

Let us assume that a change in elasticity module by a radius occurs according to the linear law:

$$G(r)=G_* r, \lambda(r)=\lambda_* r,$$

where G_*, λ_* are some constant magnitudes.

Equations of equilibrium in displacements take the form:

$$(L_0+\partial_1 L_1+\partial_1^2 L_2)\bar{u}=\bar{0}. \quad (1)$$

Here $\bar{u}=\bar{u}(\rho,\xi)=(u_\rho(\rho,\xi),u_\xi(\rho,\xi))^{\mathrm{T}}$, L_k is the matrix of differential operations of the following form:

$$L_0=\begin{Vmatrix}(2G_0+\lambda_0)(\partial^2+\varepsilon\partial)-2G_0\varepsilon^2 & 0\\ 0 & G_0(\partial^2+\varepsilon\partial)\end{Vmatrix},$$

$$L_1=\begin{Vmatrix}0 & e^{\varepsilon\rho}\left[\varepsilon(G_0+\lambda_0)\partial+\varepsilon^2\lambda_0\right]\\ e^{\varepsilon\rho}\left[\varepsilon^2(2G_0+\lambda_0)+\varepsilon(G_0+\lambda_0)\partial\right] & 0\end{Vmatrix},$$

$$L_2 = \left\| \begin{matrix} \varepsilon^2 G_0 e^{\varepsilon\rho} & 0 \\ 0 & (2G_0+\lambda_0)\varepsilon^2 e^{\varepsilon\rho} \end{matrix} \right\|,$$

$\partial_1 = \dfrac{\partial}{\partial\xi}$; $\partial_1^2 = \dfrac{\partial^2}{\partial\xi^2}$; $\partial = \dfrac{\partial}{\partial\rho}$; $\rho = \dfrac{1}{\varepsilon}\ln\left(\dfrac{r}{r_0}\right)$, $\xi = \dfrac{z}{r_0}$ are the new dimensionless variables; $\varepsilon = \dfrac{1}{2}\ln\left(\dfrac{r_2}{r_1}\right)$ is the small parameter, characterizing the thickness of a cylinder; $r_0 = \sqrt{r_1 r_2}$, $\xi \in [-l;l]$, $\rho \in [-1;1]$, $l = \dfrac{L}{r_0}$; $\lambda_0 = \dfrac{\lambda_* r_0}{G_1}$, $G_0 = \dfrac{G_* r_0}{G_1}$ are the dimensionless magnitudes and G_1 is some characteristic parameter, having the dimensionality of the shift module.

Suppose that the lateral side of a cylinder is free from stresses:

$$\bar{\sigma}\big|_{\rho=\pm1} = M\bar{u}\big|_{\rho=\pm1} = \bar{0}, \tag{2}$$

where

$$\bar{\sigma} = \left(\sigma_{\rho\rho}, \sigma_{\rho\xi}\right)^T, \; M = M_0 + \partial_1 M_1,$$

$$M_0 = \left\| \begin{matrix} \dfrac{(2G_0+\lambda_0)}{\varepsilon}\partial + \lambda_0 & 0 \\ 0 & \dfrac{G_0}{\varepsilon}\partial \end{matrix} \right\|,$$

$$M_1 = \left\| \begin{matrix} 0 & \lambda_0 e^{\varepsilon\rho} \\ G_0 e^{\varepsilon\rho} & 0 \end{matrix} \right\|.$$

Let us assume that boundary conditions are assigned at the ends of a cylinder:

$$\sigma_{\rho\xi}\big|_{\xi=\pm l} = f_{1s}(\rho), \quad \sigma_{\xi\xi}\big|_{\xi=\pm l} = f_{2s}(\rho). \tag{3}$$

Here $f_{1s}(\rho)$, $f_{2s}(\rho)$ $(s=1,2)$ are smooth enough functions, meeting the conditions of equilibrium.

5. Construction of homogeneous solutions for a radially non-homogeneous cylinder of small thickness

We will find solution (1), (2) in the form:

$$\bar{u}(\rho,\xi) = \bar{a}(\rho)e^{\alpha\xi}, \tag{4}$$

where

$$\bar{a}(\rho) = (u(\rho), w(\rho))^T.$$

Substituting (4) in (1), (2), we obtain:

$$\begin{cases} (L_0 + \alpha L_1 + \alpha^2 L_2)\bar{a} = \bar{0}, \\ (M_0 + \alpha M_1)\bar{a}\big|_{\rho=\pm1} = \bar{0}. \end{cases} \tag{5}$$

Applying the method of asymptotic integration of equations from the theory of elasticity [10, 11], based on three iterative processes at $\varepsilon \to 0$ for (5), we obtain three groups of solutions:

$$\text{I. } u_\rho^{(1)} = \frac{-\lambda_0}{2(G_0+\lambda_0)}Ce^{\varepsilon\rho}, \; u_\xi^{(1)} = C\xi. \tag{6}$$

These solutions correspond to two-fold eigenvalues $\alpha = 0$.

The stress corresponding to the solution (6) takes the following form:

$$\sigma_{\rho\rho}^{(1)} = \sigma_{\varphi\varphi}^{(1)} = \sigma_{\rho\xi}^{(1)} = 0, \; \sigma_{\xi\xi}^{(1)} = Cg_0 e^{\varepsilon\rho}, \tag{7}$$

where

$$g_0 = \frac{G_0(2G_0+3\lambda_0)}{G_0+\lambda_0}.$$

Net stretching along the axis of the cylinder corresponds to eigenvalue $\alpha = 0$ [4].

II. $\alpha_j = \varepsilon^{-1/2}(\alpha_{0j} + \varepsilon\alpha_{1j} + \ldots)$.

$$u_\rho^{(2)}(\rho,\xi) = \sum_{j=1}^{4} D_j U_{\rho j}^{(2)}(\rho,\xi), \; u_\xi^{(2)}(\rho,\xi) = \\ = \varepsilon^{1/2}\sum_{j=1}^{4} D_j U_{\xi j}^{(2)}(\rho,\xi), \tag{8}$$

where

$$U_{\rho j}^{(2)}(\rho,\xi) = (G_0+\lambda_0)\times \\ \times\left\{2 + \varepsilon g_1\left[\alpha_{0j}^2\rho^2 + \left(\frac{2g_1}{1+g_1}\alpha_{0j}^2 - 2\right)\rho\right] + O(\varepsilon^2)\right\}\times \\ \times\exp\left(\frac{1}{\sqrt{\varepsilon}}\left(\alpha_{0j} + \varepsilon\alpha_{1j} + \ldots\right)\xi\right),$$

$$U_{\xi j}^{(2)}(\rho,\xi) = (G_0+\lambda_0)\left\{-\alpha_{0j}\left(2\rho + \frac{2g_1}{1+g_1}\right) + \right. \\ + \varepsilon\left[\frac{(2+3g_1)}{3}\alpha_{0j}^3\rho^3 + \left(\begin{matrix} g_3\alpha_{0j} - (1+3g_1)\alpha_{0j}^2 - \\ -\dfrac{g_1^2}{1+g_1}\alpha_{0j}^3 \end{matrix}\right)\rho^2 + \right. \\ \left. \left. + \left(2g_3\alpha_{0j} - 2\alpha_{1j} - 4g_1\alpha_{0j} - 2(1+g_1)\alpha_{0j}^3\right)\rho\right] + O(\varepsilon^2)\right\}\times \\ \times\exp\left(\frac{1}{\sqrt{\varepsilon}}\left(\alpha_{0j} + \varepsilon\alpha_{1j} + \ldots\right)\xi\right),$$

$$g_1 = \frac{\lambda_0}{2G_0+\lambda_0}, \quad g_2 = \frac{4G_0(G_0+\lambda_0)}{2G_0+\lambda_0}, \quad g_3 = \frac{2G_0\lambda_0}{2G_0+\lambda_0}.$$

To determine α_{0j}, we obtain bi-quadratic equation:

$$\alpha_{0j}^4 - 3\frac{g_1^2}{(1+g_1)^2}\alpha_{0j}^2 + 3 = 0. \tag{9}$$

The stresses, corresponding to solutions (8), take the form:

$$\sigma_{\rho\rho}^{(2)} = \varepsilon(G_0+\lambda_0)\sum_{j=1}^{4} D_j\left\{\left[2g_2 + \frac{2(1-g_1)}{1+g_1}\lambda_0\alpha_{0j}^2\right](\rho+1) - \right. \\ \left. -\frac{g_2\alpha_{0j}^4}{3}(\rho^3 - 3\rho - 2) - g_3\alpha_{0j}^2(\rho+1)^2 + O(\varepsilon)\right\}\times \\ \times\exp\left(\frac{1}{\sqrt{\varepsilon}}\left(\alpha_{0j} + \varepsilon\alpha_{1j} + \ldots\right)\xi\right),$$

$$\sigma_{\rho\xi}^{(2)} = \varepsilon^{1/2}(G_0+\lambda_0)\sum_{j=1}^{4} D_j\left\{g_2\alpha_{0j}^3(\rho^2-1)+O(\varepsilon)\right\}\times$$
$$\times\exp\left(\frac{1}{\sqrt{\varepsilon}}(\alpha_{0j}+\varepsilon\alpha_{1j}+...)\xi\right), \quad (10)$$

$$\sigma_{\xi\xi}^{(2)} = (G_0+\lambda_0)\sum_{j=1}^{4} D_j\left\{2g_3(1-\alpha_{0j}^2)-2g_2\alpha_{0j}^2\rho+O(\varepsilon)\right\}\times$$
$$\times\exp\left(\frac{1}{\sqrt{\varepsilon}}(\alpha_{0j}+\varepsilon\alpha_{1j}+...)\xi\right),$$

$$\sigma_{\varphi\varphi}^{(2)} = (G_0+\lambda_0)\sum_{j=1}^{4} D_j\left\{\begin{matrix}2g_2-\dfrac{2g_1(1-g_1)}{1+g_1}\lambda_0\alpha_{0j}^2-\\ -2g_3\alpha_{0j}^2\rho+O(\varepsilon)\end{matrix}\right\}\times$$
$$\times\exp\left(\frac{1}{\sqrt{\varepsilon}}(\alpha_{0j}+\varepsilon\alpha_{1j}+...)\xi\right).$$

III. 1) $\alpha_k=\varepsilon^{-1}(\beta_{0k}+\varepsilon\beta_{1k}+...)$.

$$u_\rho^{(3,1)}(\rho,\xi)=\varepsilon\sum_{k=1}^{\infty}T_k\left[\left(2\beta_{0k}^2\cos\beta_k+\frac{4}{1+g_1}\beta_{0k}\sin\beta_{0k}\right)\times\right.$$
$$\left.\times\sin(\beta_{0k}\rho)-2\beta_{0k}^2\rho\sin\beta_{0k}\cos(\beta_{0k}\rho)\ +O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\sqrt{\varepsilon}}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right), \quad (11)$$

$$u_\xi^{(3,1)}(\rho,\xi)=\varepsilon\sum_{k=1}^{\infty}T_k\left[\left(\frac{2(1-g_1)}{1+g_1}\beta_{0k}\sin\beta_{0k}-2\beta_{0k}^2\cos\beta_{0k}\right)\times\right.$$
$$\left.\times\cos(\beta_{0k}\rho)-2\beta_{0k}^2\rho\sin\beta_{0k}\sin(\beta_{0k}\rho)\ +O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\sqrt{\varepsilon}}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right).$$

Here, β_{0k} is the solution to equation:

$$\sin 2\beta_{0k}+2\beta_{0k}=0. \quad (12)$$

The stresses, corresponding to solutions (11), take the form:

$$\sigma_{\rho\rho}^{(3,1)}=\sum_{k=1}^{\infty}4G_0\beta_{0k}^2T_k\left[(\beta_{0k}\cos\beta_{0k}+\sin\beta_{0k})\cos(\beta_{0k}\rho)+\right.$$
$$\left.+\beta_{0k}\rho\sin\beta_{0k}\sin(\beta_{0k}\rho)+O(\varepsilon)\right]\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right),$$

$$\sigma_{\rho\xi}^{(3,1)}=\sum_{k=1}^{\infty}4G_0\beta_{0k}^3T_k\begin{bmatrix}(\cos\beta_{0k}\sin(\beta_{0k}\rho)-\\ -\rho\sin\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\end{bmatrix}\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right),$$

$$\sigma_{\varphi\varphi}^{(3,1)}=\sum_{k=1}^{\infty}2g_3\beta_{0k}^2T_k\left[\sin\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right),$$

$$\sigma_{\xi\xi}^{(3,1)}=\sum_{k=1}^{\infty}4G_0\beta_{0k}^2T_k\begin{bmatrix}\sin\beta_{0k}\cos(\beta_{0k}\rho)-\\ -\beta_{0k}\rho\sin\beta_{0k}\sin(\beta_{0k}\rho)-\\ -\beta_{0k}\cos\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\end{bmatrix}\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right). \quad (13)$$

2) $\alpha_k=\varepsilon^{-1}(\beta_{0k}+\varepsilon\beta_{1k}+...)$.

$$u_\rho^{(3,2)}=\varepsilon\sum_{k=1}^{\infty}F_k\left[\left(2\beta_{0k}^2\sin\beta_{0k}-\frac{4}{1+g_1}\beta_{0k}\cos\beta_{0k}\right)\times\right.$$
$$\left.\times\cos(\beta_{0k}\rho)-2\beta_{0k}^2\rho\cos\beta_{0k}\sin(\beta_{0k}\rho)\ +O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right), \quad (14)$$

$$u_\xi^{(3,2)}=\varepsilon\sum_{k=1}^{\infty}F_k\left[2\beta_{0k}^2\rho\cos\beta_{0k}\cos(\beta_{0k}\rho)+\right.$$
$$\left.+\left(\frac{2(1-g_1)}{1+g_1}\beta_{0k}\cos\beta_{0k}+2\beta_{0k}^2\sin\beta_{0k}\right)\times\sin(\beta_{0k}\rho)+O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right).$$

Here β_{0k} is the solution to the equation:

$$\sin 2\beta_{0k}-2\beta_{0k}=0. \quad (15)$$

The stresses, corresponding to solutions (14), take the form:

$$\sigma_{\rho\rho}^{(3,2)}=-\sum_{k=1}^{\infty}4F_kG_0\beta_{0k}^2\left[(\beta_{0k}\sin\beta_{0k}+\cos\beta_{0k})\sin(\beta_{0k}\rho)+\right.$$
$$\left.+\beta_{0k}\rho\cos\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\right]\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right),$$

$$\sigma_{\rho\xi}^{(3,2)}=\sum_{k=1}^{\infty}4F_k\beta_{0k}^3G_0\begin{bmatrix}\sin\beta_{0k}\cos(\beta_{0k}\rho)-\\ -\rho\cos\beta_{0k}\sin(\beta_{0k}\rho)+O(\varepsilon)\end{bmatrix}\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right), \quad (16)$$

$$\sigma_{\varphi\varphi}^{(3,2)}=\sum_{k=1}^{\infty}2F_kg_3\beta_{0k}^2\left[\cos\beta_{0k}\sin(\beta_{0k}\rho)+O(\varepsilon)\right]\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right),$$

$$\sigma_{\xi\xi}^{(3,2)}=\sum_{k=1}^{\infty}4F_kG_0\beta_{0k}^2\begin{bmatrix}\beta_{0k}\sin\beta_{0k}\sin(\beta_{0k}\rho)+\\ +\cos\beta_{0k}\sin(\beta_{0k}\rho)+\\ +\beta_{0k}\rho\cos\beta_{0k}\cos(\beta_{0k}\rho)+O(\varepsilon)\end{bmatrix}\times$$
$$\times\exp\left(\frac{1}{\varepsilon}(\beta_{0k}+\varepsilon\beta_{1k}+...)\xi\right).$$

The general solution (5) will be the sum of solutions (6), (8), (11), (14), corresponding to the above three iterative processes:

$$u_\rho(\rho,\xi)=u_\rho^{(1)}+u_\rho^{(2)}+u_\rho^{(3,1)}+u_\rho^{(3,2)},$$

$$u_\xi(\rho,\xi)=u_\xi^{(1)}+u_\xi^{(2)}+u_\xi^{(3,1)}+u_\xi^{(3,2)}. \quad (17)$$

For the component of tensor of stress, we obtain:

$$\sigma_{\rho\xi}=\sigma_{\rho\xi}^{(2)}+\sigma_{\rho\xi}^{(3,1)}+\sigma_{\rho\xi}^{(3,2)},$$

$$\sigma_{\xi\xi}=\sigma_{\xi\xi}^{(1)}+\sigma_{\xi\xi}^{(2)}+\sigma_{\xi\xi}^{(3,1)}+\sigma_{\xi\xi}^{(3,2)}, \quad (18)$$

$$\sigma_{\rho\rho}=\sigma_{\rho\rho}^{(2)}+\sigma_{\rho\rho}^{(3,1)}+\sigma_{\rho\rho}^{(3,2)},$$

$$\sigma_{\varphi\varphi}=\sigma_{\varphi\varphi}^{(2)}+\sigma_{\varphi\varphi}^{(3,1)}+\sigma_{\varphi\varphi}^{(3,2)}. \quad (19)$$

6. Analysis of the stressed-strained state determined by homogeneous solutions

Displacements will be represented in the form:

$$u_\rho(\rho,\xi)=u_\rho^{(1)}+\sum_{k=1}^{\infty}E_k u_k(\rho)e^{\alpha_k\xi},$$

$$u_\xi(\rho,\xi)=u_\xi^{(1)}+\sum_{k=1}^{\infty}E_k w_k(\rho)e^{\alpha_k\xi}. \quad (20)$$

The second term includes the displacement, determined by the second and third iterative process.

For stresses, we obtain:

$$\sigma_{\xi\xi}=\sigma_{\xi\xi}^{(1)}+\sum_{k=1}^{\infty}E_k\sigma_{2k}(\rho)e^{\alpha_k\xi},\quad \sigma_{\rho\xi}=\sum_{k=1}^{\infty}E_k\sigma_{1k}(\rho)e^{\alpha_k\xi}, \quad (21)$$

where

$$\sigma_{1k}(\rho)=\frac{G_0}{\varepsilon}\left(w_k'(\rho)+\varepsilon\alpha_k e^{\varepsilon\rho}u_k(\rho)\right),$$

$$\sigma_{2k}(\rho)=\frac{1}{\varepsilon}\left[(2G_0+\lambda_0)\varepsilon\alpha_k e^{\varepsilon\rho}w_k(\rho)+\lambda_0(u_k'(\rho)+\varepsilon u_k(\rho))\right].$$

Consider the relationship of the constructed solutions with the main vector P of stresses acting in cross section $\xi=\text{const}$. It should be noted that:

$$P=2\pi\varepsilon\int_{-1}^{1}\left(\sigma_{\xi\xi}+\sigma_{\rho\xi}\right)e^{2\varepsilon\rho}d\rho. \quad (22)$$

Substituting (21) in (22), we will obtain:

$$P=\frac{4\pi}{3}g_0 Csh(3\varepsilon)+2\pi\varepsilon\sum_{k=1}^{\infty}E_k m_k e^{\alpha_k\xi}, \quad (23)$$

where

$$m_k=\int_{-1}^{1}\left(\sigma_{1k}(\rho)+\sigma_{2k}(\rho)\right)e^{2\varepsilon\rho}d\rho.$$

We will prove that all $m_k=0$ $(k=1,2\ldots)$. To do this, we will consider the following boundary-value problem:

$$\sigma_{\rho\xi}=\sigma_{1k}(\rho)e^{\alpha_k\xi_j},\ \sigma_{\xi\xi}=\sigma_{2k}(\rho)e^{\alpha_k\xi_j}\ \text{at}\ \xi=\xi_j\ (j=1,2). \quad (24)$$

The main vector, which corresponds to the stressed state of problem (24) in cross-section $\xi=\text{const}$, is reduced to the following form:

$$P_k=2\pi\varepsilon m_k e^{\alpha_k\xi}. \quad (25)$$

According to the solvability of the problem of the elasticity theory, P_k should not depend on variable ξ. However, in (25) the right part depends on ξ. Hence, it follows that $P_k=0$, i. e. $m_k=0$. From (23), we obtain:

$$P=\frac{4\pi}{3}g_0 Csh(3\varepsilon). \quad (26)$$

The stressed state that corresponds to the second and third groups of solutions is self-balanced in any cross-section $\xi=\text{const}$.

Solutions (6), (8) determine the internal stressed-strained state of a cylinder. The first terms of their asymptotic decompositions for parameter ε can be considered as a solution by the applied theory of shells. Solution (6) that corresponds to the first iterative process penetrates without attenuation inside the area of a cylinder and is a penetrating solution. Solution (8) determines the edge effect similar to a simple edge effect in the applied theory of shells [1, 3, 4, 11]. Solution (11), (14) has the character of a boundary layer. The first terms of its asymptotic decomposition are equivalent to the Saint-Venant edge effect for a non-homogeneous plate [1].

The stresses, determined from formulas (10), (13), (16), that is, by the second and third iterative processes, are localized at the ends of a cylinder and exponentially decrease at removing from the ends. The indicators of attenuation of stresses, determined by the second iterative processes, have order $O(\varepsilon^{-1/2})$, relative to ε, and stresses corresponding to the third iterative process, have order $O(\varepsilon^{-1})$.

As it was shown, the non-self-balanced part of stresses can be removed with the help of the penetrating solution (6), and in this case, the relation of constant C with the main vector P is assigned by equality (26).

7. Meeting boundary conditions at the ends of a cylinder

To determine the unknown constants D_j $(j=\overline{1,4})$, T_k and F_k $(k=1,2,\ldots)$, we will use the Lagrangian variation principle. Since solutions (17) satisfy the equilibrium equation and boundary conditions on the lateral surface, the variation principle takes the following form [2, 3, 12]:

$$\sum_{s=1}^{2}\int_{-1}^{1}\left[\left(\sigma_{\rho\xi}-f_{1s}\right)\delta u_\rho+\left(\sigma_{\xi\xi}-f_{2s}\right)\delta u_\xi\right]\Bigg|_{\xi=\pm l}e^{2\varepsilon\rho}d\rho=0. \quad (27)$$

Substituting (17), (18) in (27) and considering δD_j, δT_k, δF_k to be independent variations, we obtain from (27) the following systems of linear algebraic equations:

$$\sum_{j=1}^{4}m_{kj}D_{j0}=\tau_k;\ \ (k=\overline{1,4}), \quad (28)$$

$$\sum_{k=1}^{\infty}M_{jk}T_{k0}=d_{0j}';\ \ (j=\overline{1,\infty}), \quad (29)$$

$$\sum_{k=1}^{\infty}Q_{jk}F_{k0}=d_{0j}'';\ \ (j=\overline{1,\infty}), \quad (30)$$

where

$$
\begin{aligned}
m_{kj} = {} & \frac{1}{6(G_0+\lambda_0)}\Big[-32G_0(G_0+\lambda_0)^3\alpha_{0j}^3 + 8G_0(G_0+\lambda_0)\times \\
& \times(4G_0^2+8G_0\lambda_0+7\lambda_0^2)\alpha_{0j}^2\alpha_{0k} - 24G_0\lambda_0^2(G_0+\lambda_0)\alpha_{0k}\Big]\times \\
& \times\left(\exp\left(-\frac{(\alpha_{0j}+\alpha_{0k})l}{\sqrt{\varepsilon}}\right)+\exp\left(\frac{(\alpha_{0j}+\alpha_{0k})l}{\sqrt{\varepsilon}}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\tau_k = {} & \int_{-1}^{1}\left\{\begin{bmatrix} 2(G_0+\lambda_0)\gamma_1 - \\ \alpha_{0k}(2(G_0+\lambda_0)\rho+\lambda_0)f_{21}\end{bmatrix}\left(\exp\left(-\frac{\alpha_{0k}l}{\sqrt{\varepsilon}}\right)+\right.\right. \\
& \left.\left.+\begin{bmatrix} 2(G_0+\lambda_0)\gamma_2 - \\ -\alpha_{0k}(2(G_0+\lambda_0)\rho+\lambda_0)f_{22}\end{bmatrix}\exp\left(\frac{\alpha_{0k}l}{\sqrt{\varepsilon}}\right)\right)\right\}d\rho,
\end{aligned}
$$

$$
\begin{aligned}
M_{jk} = {} & \int_{-1}^{1} 4G_0\beta_{0k}^2\left\{\beta_{0k}\begin{bmatrix}\cos\beta_{0k}\sin(\beta_{0k}\rho)- \\ -\rho\sin\beta_{0k}\cos(\beta_{0k}\rho)\end{bmatrix}\times\right. \\
& \times\begin{bmatrix}\left(2\beta_{0j}\cos\beta_{0j}+\dfrac{2(2G_0+\lambda_0)}{G_0+\lambda_0}\sin\beta_{0j}\right)\times \\ \times\sin(\beta_{0j}\rho)-2\beta_{0j}\rho\sin\beta_{0j}\cos(\beta_{0j}\rho)\end{bmatrix}+ \\
& +\begin{bmatrix}\sin\beta_{0k}\cos(\beta_{0k}\rho)-\beta_{0k}\rho\sin\beta_{0k}\sin(\beta_{0k}\rho)- \\ -\beta_{0k}\cos\beta_{0k}\cos(\beta_{0k}\rho)\end{bmatrix}\times \\
& \left.\times\begin{bmatrix}\left(\dfrac{2G_0}{G_0+\lambda_0}\sin\beta_{0j}-2\beta_{0j}\cos\beta_{0j}\right)\cos(\beta_{0j}\rho)- \\ -2\beta_{0j}\rho\sin\beta_{0j}\sin(\beta_{0j}\rho)\end{bmatrix}\right\}d\rho\times \\
& \times\left[\exp\left(-\frac{(\beta_{0k}+\beta_{0j})l}{\varepsilon}\right)+\exp\left(\frac{(\beta_{0k}+\beta_{0j})l}{\varepsilon}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
d'_{0j} = {} & \int_{-1}^{1}\sum_{s=1}^{2}\left\{f_{1s}(\rho)\begin{bmatrix}\left(2\beta_{0j}\cos\beta_{0j}+\dfrac{2(2G_0+\lambda_0)}{G_0+\lambda_0}\sin\beta_{0j}\right)\times \\ \times\sin(\beta_{0j}\rho)-2\beta_{0j}\rho\sin\beta_{0j}\cos(\beta_{0j}\rho)\end{bmatrix}+\right. \\
& \left.+f_{2s}^*(\rho)\begin{bmatrix}\left(\dfrac{2G_0}{G_0+\lambda_0}\sin\beta_{0j}-2\beta_{0j}\cos\beta_{0j}\right)\times \\ \times\cos(\beta_{0j}\rho)-2\beta_{0j}\rho\sin\beta_{0j}\sin(\beta_{0j}\rho)\end{bmatrix}\right\}dp\times \\
& \times\exp\left((-1)^s\frac{\beta_{0j}l}{\varepsilon}\right),
\end{aligned}
$$

$$
\begin{aligned}
Q_{jk} = {} & \int_{-1}^{1}4G_0\beta_{0k}^2\left\{\beta_{0k}\left[\sin\beta_{0k}\cos(\beta_{0k}\rho)-\rho\cos\beta_{0k}\sin(\beta_{0k}\rho)\right]\times\right. \\
& \times\begin{bmatrix}\left(2\beta_{0j}\sin\beta_{0j}-\dfrac{2(2G_0+\lambda_0)}{G_0+\lambda_0}\cos\beta_{0j}\right)\cos(\beta_{0j}\rho)- \\ -2\beta_{0j}\rho\cos\beta_{0j}\sin(\beta_{0j}\rho)\end{bmatrix}+ \\
& +\begin{bmatrix}\beta_{0k}\sin\beta_{0k}\sin(\beta_{0k}\rho)+\cos\beta_{0k}\sin(\beta_{0k}\rho)+ \\ +\beta_{0k}\rho\cos\beta_{0k}\cos(\beta_{0k}\rho)\end{bmatrix}\times \\
& \left.\times\begin{bmatrix}2\beta_{0j}\rho\cos\beta_{0j}\cos(\beta_{0j}\rho)+ \\ +\left(\dfrac{2G_0}{G_0+\lambda_0}\cos\beta_{0j}+2\beta_{0j}\sin\beta_{0j}\right)\sin(\beta_{0j}\rho)\end{bmatrix}\right\}d\rho\times \\
& \times\left(\exp\left(\frac{-(\beta_{0k}+\beta_{0j})l}{\varepsilon}\right)+\exp\left(\frac{(\beta_{0k}+\beta_{0j})l}{\varepsilon}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
d''_{0j} = {} & \int_{-1}^{1}\sum_{s=1}^{2}\left\{f_{1s}(\rho)\begin{bmatrix}\left(2\beta_{0j}\sin\beta_{0j}-\dfrac{2(2G_0+\lambda_0)}{G_0+\lambda_0}\cos\beta_{0j}\right)\times \\ \times\cos(\beta_{0j}\rho)-2\beta_{0j}\rho\cos\beta_{0j}\sin(\beta_{0j}\rho)\end{bmatrix}+\right. \\
& \left.+f_{2s}^*(\rho)\begin{bmatrix}2\beta_{0j}\rho\cos\beta_{0j}\cos(\beta_{0j}\rho)+ \\ +\left(\dfrac{2G_0}{G_0+\lambda_0}\cos\beta_{0j}+2\beta_{0j}\sin\beta_{0j}\right)\sin(\beta_{0j}\rho)\end{bmatrix}\right\}d\rho\times \\
& \times\exp\left(\frac{(-1)^s\beta_{0j}l}{\varepsilon}\right),
\end{aligned}
$$

$$
f_{1s}^{(1)}=\frac{1}{2}\int_{-1}^{1}f_{1s}(\rho)d\rho,\quad f_{1s}(\rho)=f_{1s}^{(1)}+f_{1s}^{(2)},\quad \gamma_s=\lim_{\varepsilon\to 0}\frac{f_{1s}^{(1)}}{\sqrt{\varepsilon}},
$$

$$
f_{2s}^*(\rho)=f_{2s}(\rho)-\frac{3Pe^{\varepsilon\rho}}{4\pi\, sh3\varepsilon},\quad D_j=D_{j0}+\varepsilon D_{j1}+\ldots,
$$

$$
T_k=T_{0k}+\varepsilon T_{k1}+\ldots,\ F_k=F_{0k}+\varepsilon F_{k1}+\ldots.
$$

Determining constants D_{jp}, T_{kp}, F_{kp} $(p=1,2,\ldots)$ is reduced to the systems, the matrices of which coincides with matrices of systems (28)–(30).

The system of infinite linear algebraic equations (29), (30) is positively determined in the space of energy H, and that is why it is always solvable under physically meaningful conditions imposed on the right side [4]. Solvability and convergence of the method for reduction for (29), (30) was proved in [1, 13].

8. Discussion of results of studying a three-dimensional problem of elasticity theory for a radially non-homogeneous cylinder of small thickness

Using the method of asymptotic integration of the equations of the elasticity theory, the homogeneous solutions were constructed. Based on the conducted analysis, it was obtained that homogeneous solutions include three types of solutions: penetrating solutions, solution of the type of simple edge effect and solution having the nature of the boundary layer.

The character of the constructed homogeneous solutions was explained. The solutions corresponding to the first iterative process penetrate without attenuation inside a cylinder and are a penetrating solution. The first terms of their decomposition by the parameter of wall thinness of a cylinder determine the momentless stressed state. The solutions corresponding to the second iteration process represent the edge effects in the applied theory of shells. In the first terms of decomposition by the parameter of wall thinness of a cylinder, the penetrating solution, together with the edge effect solution, can be considered as a solution by the applied theory of shells. The solutions corresponding to the third iterative process have the character of a boundary layer and do not exist in applied theories. The stresses determined by the second and third iterative processes are localized at the ends of a cylinder and at remoteness from the ends, exponentially damp at different rates.

Using the Lagrange principle of possible displacements, the problem of meeting the boundary conditions at the ends of a cylinder was explored. The non-self-balanced load part, assigned at the ends of a cylinder is removed using a penetrating solution. To find the constants included in the solution, determined by the second and third iterative processes, respectively, the finite and non-finite systems of linear algebraic equations were obtained.

The obtained asymptotic formulas make it possible to calculate the three-dimensional stressed-strained state of a radially non-homogeneous cylinder of small thickness under various boundary conditions on the ends of the cylinder. The constructed solutions are the basis for the assessment of existing theories. Using the constructed solutions, it is possible to suggest a new refined applied theory, which more accurately describes the processes that occur in a radially non-homogeneous cylinder of small thickness.

The obtained asymptotic formulas are only suitable for a radially non-homogeneous cylinder of small thickness. For a thick radially non-homogeneous cylinder, the separation of the stressed-strained state into the internal and boundary-layer solutions are impossible. It is characteristic only for thin cylinders.

The phenomenon of a weak boundary layer is absent in small non-homogeneous cylinders of small thickness (a cylinder, where the values of elasticity modules vary within a single order). In this case, the method for asymptotic integration of the equations of the theory of elasticity makes it possible to solve the axisymmetric problem of the elasticity theory for radially non-homogeneous cylinder of small thickness. For a radially-layered cylinder of small thickness with alternating hard and soft layers (strongly heterogeneous cylinder), there appear two different small parameters: small parameter characterizing the thickness of a cylinder and small parameter, which is responsible for the relative characteristic of layers' stiffness. The asymptotic behavior of the solution depends on the ratio of these small parameters. As a result of the collision of two small parameters, a weak boundary layer appears. The processes of determining the internal solution and a weak boundary layer are not separated. That is why in the case of a strongly non-homogeneous cylinder of small thickness, it is convenient to apply the disturbance methods of the theory of operators.

9. Conclusions

1. The asymptotic analysis revealed the features of the stressed-strained state in a radially non-homogeneous cylinder. The analysis identified three groups of solutions. The solution, corresponding to the first iterative process, determines the penetrating stressed-strained state of a cylinder. The stressed state, determined by this solution, is equivalent of the main vector of forces applied in an arbitrary cross-section $\xi = \text{const}$ of a cylinder. The stressed state, corresponding to the second iterative process, represents edge effects in the applied theory of shells. The third iterative process determines the solutions, which have the character of a boundary layer and are localized at the ends of a cylinder.

2. It was shown that the solution that corresponds to the first and second iterative processes determines the internal stressed-strained state of a cylinder. It was proved that stressed state that corresponds to the second and third groups of solutions is self-balanced in each section $\xi = \text{const}$.

3. The new groups of solutions corresponding to the third iterative process, which are not found in the applied theories, were found. The asymptotic formulas for displacement and stresses were obtained.

4. The problem of meeting boundary conditions at the ends of a cylinder using the Lagrangian variation principle was considered.

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