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Asymptotic analysis of three-dimensional problem of elasticity theory for radially inhomogeneous transversally-isotropic thin hollow spheres

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ABSTRACT

In this study, three-dimensional problem of the theory of elasticity (3DPTE) for radially inhomogeneous (INH) transversally-isotropic thin hollow spheres is investigated using the asymptotic integration method. The basic relations and equilibrium equations for radially inhomogeneous transversally-isotropic thin hollow spheres are formed and inhomogeneous solutions (INHSs) and homogeneous solutions (HSs) are constructed. The built solutions completely reveal the qualitative structure of a three-dimensional stress-strain state of radially inhomogeneous transversely-isotropic spheres of small thickness and serve as an effective apparatus for solving boundary value problems, the basis for evaluating existing applied theories and for creating new, more refined applied theories. Asymptotic formulas are obtained that allow the calculation of the three-dimensional stress-strain state of spheres. New solution groups (boundary layer solutions) have been found that are absent in applied theories. The behavior of homogeneous solutions in the inner parts of the sphere and in the vicinity of conical sections has been studied, when the thin-walled parameter of the sphere tends to zero. The nature of the constructed homogeneous solutions is clarified. On the basis of the theoretical analysis, three types of the stress-strain state (SSS) in the radially inhomogeneous transversally-isotropic hollow spheres (RINHTIHSs) are considered: a penetrating stress state, a simple edge effect, and a boundary layer. Finally numerical calculations are made and the influences of inhomogeneity on the stress distributions are investigated.

1. Introduction

Since the spherical shells are one of the basic elements of a series of technical designs, the study of three-dimensional problems of their elasticity theory has created the subject of many works [1,2]. The problem of elasticity theory for the spherical shells is very old and is considered by Saint-Venant. The first attempts at solving the equilibrium problems of elastic spherical shells have been made by Galerkin [3], Lurie [4] and Lekhnitsky [5]. Then Gol'denveizer [6] derived an approximate theory of shells by means of asymptotic integration for the equations of the elasticity theory.

One of the first attempts to solve the elasticity problem of the sphere using the asymptotic method was done by Vilenskaya and Vorovich [7]. After this research, some important studies emerged; Rappoport [8] investigated the deformations of isotropic and transversally-isotropic thick spheres. Vasilenko et al. [9] analyzed the three-dimensional stress-strain state for INH transversely isotropic spheres. Boev and Ustinov [10] carried out an analysis of the three-dimensional SSS of three-layered spheres. Cimetière et al. [11] developed asymptotic theory and analysis for displacements and stress distributions in nonlinear elastic straight slender rods. Ciarlet et al. [12] presented asymptotic analysis of linearly elastic shells, justification for flexural shell equations. It is also necessary to emphasize some important publications on the 3-D exact solutions of elasticity for spherical shells by state-space-method. For example; Chen and Ding [13] examined the exact static stress analysis of multilayered elastic spherical shells completely based on three-dimensional elasticity for spherical isotropy using the state-space-method. Ding et al. [14] studied the elastodynamic solution for spherically symmetric problems of multilayered hollow spheres using the state-space-method, the method of separation of variables and the method of orthotropic expansion.

An important place in the theory of thin-walled structures is occupied by the study of the behavior of inhomogeneous shells. High specific strength and stiffness with relatively low weight, heat-insulating, electrical insulating, soundproofing properties and high aerodynamic qualities of heterogeneous structures ensure their wide application in various fields of engineering. The complexity of the phenomena that arise when deforming inhomogeneous shells and the variety of

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heterogeneity of structures led to the creation of various applied theories built on the basis of a certain system of hypotheses [15–25].

The existence of various applied theories for inhomogeneous shells requires their analysis on the basis of 3DPTE. The analysis of SSS of INH shells from the position of 3-D equations of the theory of elasticity for creating new more refined applied theories is actual [1]. In addition, a number of issues related to the study of the SSS of inhomogeneous shells, including the study of the stress concentration near the boundaries and local loads, and the study of vibrations in a sufficiently wide frequency range and in a number of other cases, can be solved only in the framework of theory of elasticity.

In the literature, there are at different methods to 3D exact solutions of elasticity for structural elements, one of these is the asymptotic method. Akhmedov and Mekhtiev [26,27] conducted asymptotic analysis of the inhomogeneous cone and plates using asymptotic method. Cheng et al. [28] applied three-dimensional asymptotic approach to inhomogeneous and laminated piezoelectric plates. Mekhtiev and Bergman [29] investigated the forced vibration of the transverse isotropic hollow cylinder under axisymmetric harmonic force based on the theory of elasticity using the homogenous solutions method. Vetyukov et al. [30] used asymptotic splitting in the three dimensional problem of elasticity for non-homogeneous piezoelectric plates. Akhmedov and Akperova [31] presented asymptotic analysis of the 3DPTE for a radially inhomogeneous transversely-isotropic hollow cylinder. Kulikov and Plotnikova [32,33] presented exact 3D stress analysis and rigid-body motions of laminated composite plates by sampling surfaces method. Shariyat and coauthors [34–38] solved 3DPTE and studied stress, bending, vibration and buckling analysis of FGM and inhomogeneous plates with and without eccentric cutouts using different methods.

The challenge of the current study is the solution of the 3-D problem of the theory of elasticity for thin RINHTIHSs by using the method of asymptotic integration of the equations of the theory of elasticity. The INHSs and HSs are created. Asymptotic formulas are obtained that allow the calculation of the three-dimensional stress-strain state of spheres. New solution groups (boundary layer solutions) have been found that are absent in applied theories. The behavior of homogeneous solutions in the inner parts of the sphere and in the vicinity of conical sections has been studied, when the thin-walled parameter of the sphere tends to zero. In addition, the nature of the HSs constructed is studied. On the basis of the analysis, it is shown that the SSS in the RINHTIHS consists of three types: a penetrating stress state, a simple edge effect, and a boundary layer. Finally numerical calculations are made and the effects of inhomogeneity of stress distributions are investigated.

2. Basic relations and equilibrium equations

Let us consider the axisymmetric problem of the theory of elasticity for a radially inhomogeneous transversely-isotropic hollow sphere of the small thickness (see, Fig. 1). We assume that the sphere does not contain any of the poles of 0 and π. In the spherical coordinate system, the region occupied by the sphere is denoted by $\Gamma = \{r \in [r_1, r_2],\ \theta \in [\theta_1, \theta_2],\ \varphi \in [0, 2\pi]\}$.

The system of equilibrium equations in the absence of mass forces in the spherical coordinate system r, φ and θ expressed as [1–4]:

$$\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{2\sigma_{rr} - \sigma_{\varphi\varphi} - \sigma_{\theta\theta} + \sigma_{r\theta}\cot\theta}{r} = 0$$
$$\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta}{r} = 0 \tag{1}$$

where σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\varphi\varphi}$, $\sigma_{\theta\theta}$ are components of the stress tensor, which are expressed in terms of the displacement components $w_r = w_r(r, \theta)$ and $w_\theta = w_\theta(r, \theta)$ as follows [5,9,18,25]:

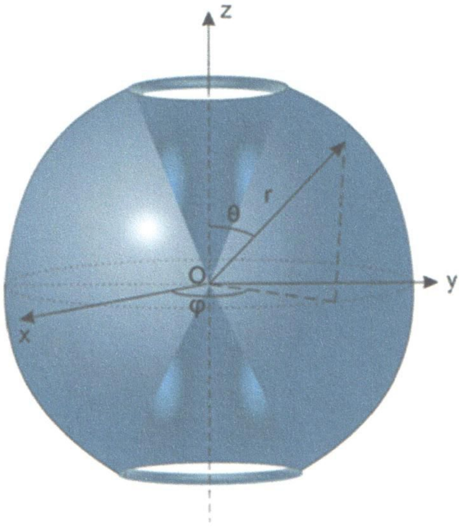

Fig. 1. Radially inhomogeneous transversely-isotropic hollow sphere and notation.

$$\sigma_{rr} = A_{11}\frac{\partial w_r}{\partial r} + A_{12}\left(\frac{w_\theta}{r}\cot\theta + \frac{2w_r}{r} + \frac{1}{r}\frac{\partial w_\theta}{\partial\theta}\right)$$
$$\sigma_{\varphi\varphi} = A_{12}\frac{\partial w_r}{\partial r} + (A_{22} + A_{23})\frac{w_r}{r} + A_{22}\frac{w_\theta}{r}\cot\theta + \frac{A_{23}}{r}\frac{\partial w_\theta}{\partial\theta}$$
$$\sigma_{\theta\theta} = A_{12}\frac{\partial w_r}{\partial r} + (A_{22} + A_{23})\frac{w_r}{r} + A_{23}\frac{w_\theta}{r}\cot\theta + \frac{A_{22}}{r}\frac{\partial w_\theta}{\partial\theta}$$
$$\sigma_{r\theta} = A_{44}\left(\frac{1}{r}\frac{\partial w_r}{\partial\theta} + \frac{\partial w_\theta}{\partial r} - \frac{w_\theta}{r}\right) \tag{2}$$

where $A_{ij} = A_{ij}(\bar{r})$ are the material properties of the RINHTIHS, regarded as arbitrary continuous functions of the variable $\bar{r}$, in which $\bar{r} = r/r_0$ and $r_0 = \sqrt{r_1 r_2}$. Such shells will be called weakly inhomogeneous.

Substituting relations (2) into the Eq. (1), after some manipulations, we obtain the equilibrium equations in the displacements as:

$$\frac{\partial}{\partial\rho}\left[e^{-\varepsilon\rho}\left(b_{11}\frac{\partial u_\rho}{\partial\rho} + \varepsilon b_{12}\left(u_\theta\cot\theta + 2u_\rho + \frac{\partial u_\theta}{\partial\theta}\right)\right)\right]$$
$$+ \varepsilon e^{-\varepsilon\rho}\left[\varepsilon(2b_{12} - b_{22} - b_{23} - b_{44})\frac{\partial u_\theta}{\partial\theta} + (2b_{12} - b_{23} - b_{44} - b_{22})\varepsilon u_\theta\cot\theta + 2(b_{11} - b_{12})\frac{\partial u_\rho}{\partial\rho} + 2(2b_{12} - b_{22} - b_{23})\varepsilon u_\rho + b_{44}\frac{\partial^2 u_\theta}{\partial\rho\partial\theta} + \varepsilon b_{44}\frac{\partial^2 u_\rho}{\partial\theta^2} + b_{44}\frac{\partial u_\theta}{\partial\rho}\cot\theta + \varepsilon b_{44}\frac{\partial u_\rho}{\partial\theta}\cot\theta\right] = 0$$
$$\frac{\partial}{\partial\rho}\left[b_{44}e^{-\varepsilon\rho}\left(\frac{\partial u_\theta}{\partial\rho} + \varepsilon\frac{\partial u_\rho}{\partial\theta} - \varepsilon u_\theta\right)\right] + \varepsilon e^{-\varepsilon\rho}\left[b_{12}\frac{\partial^2 u_\rho}{\partial\rho\partial\theta} + \varepsilon(b_{22} + b_{23} + 3b_{44}) \times \frac{\partial u_\rho}{\partial\theta} - \varepsilon(b_{23} + 3b_{44} + b_{22}\cot^2\theta)u_\theta + \varepsilon b_{22}\left(\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\partial u_\theta}{\partial\theta}\cot\theta\right) + 3b_{44}\frac{\partial u_\theta}{\partial\rho}\right] = 0 \tag{3}$$

where $\rho = \frac{1}{\varepsilon}\ln\left(\frac{r}{r_0}\right)$ is the new non-dimensional radial variable and $(-1 \le \rho \le 1)$, $\varepsilon = \frac{1}{2}\ln\left(\frac{r_2}{r_1}\right)$ is the non-dimensional small parameter characterizing the thickness of the sphere and $u_\rho = \frac{w_r}{r_0}$, $u_\theta = \frac{w_\theta}{r_0}$, $b_{ij} = \frac{A_{ij}}{G_0}$ in which G_0 is the some characteristic modulus and defined as, $G_0 = \max A_{ij}$.

Suppose that on the lateral surfaces of the sphere the following boundary conditions are given:

$$\sigma_{\rho\rho}=\frac{e^{-\varepsilon\rho}}{\varepsilon}\left[b_{11}\frac{\partial u_\rho}{\partial\rho}+\varepsilon b_{12}\left(u_\theta\cot\theta+2u_\rho+\frac{\partial u_\theta}{\partial\theta}\right)\right]\Bigg|_{\rho=\pm1}=f^{\pm}(\theta),$$
$$\sigma_{\rho\theta}=\frac{b_{44}e^{-\varepsilon\rho}}{\varepsilon}\left[\frac{\partial u_\theta}{\partial\rho}+\varepsilon\left(\frac{\partial u_\rho}{\partial\theta}-u_\theta\right)\right]\Bigg|_{\rho=\pm1}=t^{\pm}(\theta) \tag{4}$$

where the following dimensionless parameters apply: $\sigma_{\rho\rho}=\frac{\sigma_{rr}}{G_0}$ and $\sigma_{\rho\theta}=\frac{\sigma_{r\theta}}{G_0}$

We assume that the loads $f^{\pm}(\theta)$ and $t^{\pm}(\theta)$ given on the lateral surfaces are sufficiently smooth functions. Let us assume that arbitrary boundary conditions are given at the ends of the sphere (on the conical sections $\theta=\theta_j$ $(j=1,2)$), leaving the sphere in equilibrium.

3. Construction of solutions

3.1. Construction of inhomogeneous solutions

The INHSs will be called particular solutions of the equilibrium Eq. (3), which satisfy the inhomogeneous boundary conditions (4). As the thickness of the sphere is sufficiently small, and the load given on the lateral surfaces is sufficiently smooth and relative to ε is of order $O(1)$, then it is reasonable to use the first iteration process (IP) of the asymptotic method to construct the INHSs [6].

We seek INHSs in the form:

$$u_\rho=\varepsilon^{-1}\left(u_{\rho_0}+\varepsilon u_{\rho_1}+\varepsilon^2u_{\rho_2}+\ldots\right),\quad u_\theta=\varepsilon^{-1}\left(u_{\theta_0}+\varepsilon u_{\theta_1}+\varepsilon^2u_{\theta_2}+\ldots\right) \tag{5}$$

The substitution of (5) into (3) and (4) leads to a system whose sequential integration yields the relations for the expansion coefficients of (5):

$$u_{\rho0}=d_1(\theta),\quad u_{\theta0}=d_2(\theta),$$
$$u_{\rho1}=-[2d_1(\theta)+d_2(\theta)\cot\theta+d'_2(\theta)]\int_0^\rho\frac{b_{12}}{b_{11}}dx+d_3(\theta)$$
$$u_{\theta1}=\rho\ d_2(\theta)+d_4(\theta) \tag{6}$$

where the following definitions apply:

$$p_0(d_2''(\theta)+d_2'(\theta)\cot\theta)+\left(b_{23}^{(0)}-b_{22}^{(0)}-\frac{p_0}{\sin^2\theta}\right)d_2(\theta)-(3b_{44}^{(0)}+g_0)d_1'(\theta)=t(\theta)$$
$$b_{44}^{(0)}(d_1''(\theta)+d_1'(\theta)\cot\theta)-g_0(d_2'(\theta)+d_2(\theta)\cot\theta)-2g_0d_1(\theta)=-f(\theta) \tag{7}$$

in which

$$f(\theta)=f^+(\theta)-f^-(\theta),\quad t(\theta)=t^+(\theta)-t^-(\theta),$$
$$p_k=\int_{-1}^1\frac{(b_{12}^2-b_{11}b_{22})}{b_{11}}\rho^kd\rho$$
$$g_k=\int_{-1}^1\frac{(b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2)}{b_{11}}\rho^kd\rho,\quad b_{44}^{(k)}=\int_{-1}^1b_{44}\rho^kd\rho,$$
$$b_{23}^{(k)}=\int_{-1}^1b_{23}\rho^kd\rho,\quad b_{22}^{(k)}=\int_{-1}^1b_{22}\rho^kd\rho \tag{8}$$

The components of the stress tensor corresponding to solutions of (5) have the form:

$$\sigma_{\rho\rho}=(d_2'(\theta)+d_2(\theta)\cot\theta)\int_{-1}^\rho\frac{b_{23}b_{11}+b_{22}b_{11}-2b_{12}^2}{b_{11}}dx-(d_1''(\theta)+d_1'(\theta)\cot\theta)\int_{-1}^\rho b_{44}dx+d_1(\theta)\int_{-1}^\rho\frac{2(b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2)}{b_{11}}dx+f^-(\theta)+O(\varepsilon)$$
$$\sigma_{\rho\theta}=b_{44}\Bigg\{d_2'(\theta)\int_{-1}^\rho\frac{b_{12}^2-b_{11}b_{22}}{b_{11}}dx+d_2'(\theta)\cot\theta\int_{-1}^\rho\frac{b_{12}^2-b_{11}b_{22}}{b_{11}}dx-d_1'(\theta)\left[3\int_{-1}^\rho b_{44}dx+\int_{-1}^\rho\frac{b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2}{b_{11}}dx\right]+d_2(\theta)\left[\int_{-1}^\rho(b_{23}-b_{22})dx+\frac{1}{\sin^2\theta}\int_{-1}^\rho\frac{b_{11}b_{22}-b_{12}^2}{b_{11}}dx\right]+t^-(\theta)+O(\varepsilon)\Bigg\}$$
$$\sigma_{\varphi\varphi}^*=\frac{1}{\varepsilon}\left\{\frac{b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2}{b_{11}}d_1(\theta)++\frac{b_{23}b_{11}-b_{12}^2}{b_{11}}d_2'(\theta)+\frac{b_{11}b_{22}-b_{12}^2}{b_{11}}d_2(\theta)\cot\theta+O(\varepsilon)\right\}$$
$$\sigma_{\theta\theta}^*=\frac{1}{\varepsilon}\left\{\frac{b_{11}b_{22}+b_{11}b_{23}-2b_{12}^2}{b_{11}}d_1(\theta)+\frac{b_{11}b_{23}-b_{12}^2}{b_{11}}\cot\theta\cdot d_2(\theta)+\frac{b_{11}b_{22}-b_{12}^2}{b_{11}}d_2'(\theta)+O(\varepsilon)\right\} \tag{9}$$

where the following dimensionless parameters apply: $\sigma_{\varphi\varphi}^*=\frac{\sigma_{\varphi\varphi}}{G_0}$ and $\sigma_{\theta\theta}^*=\frac{\sigma_{\theta\theta}}{G_0}$

3.2. Construction of HSs

The HSs will be any solution of the equilibrium Eq. (3) that satisfies the condition of no stresses on the lateral surfaces. Let us consider the problem of constructing HSs. In relations (4), we set $f^{\pm}(\theta)=0$ and $t^{\pm}(\theta)=0$:

$$\left[b_{11}\frac{\partial u_\rho}{\partial\rho}+\varepsilon b_{12}\left(u_\theta\cot\theta+2u_\rho+\frac{\partial u_\theta}{\partial\theta}\right)\right]\Bigg|_{\rho=\pm1}=0,$$
$$b_{44}\left[\frac{\partial u_\theta}{\partial\rho}+\varepsilon\left(\frac{\partial u_\rho}{\partial\theta}-u_\theta\right)\right]\Bigg|_{\rho=\pm1}=0. \tag{10}$$

The solution of Eqs. (3) and (10) is sought as follows:

$$u_\rho(\rho,\theta)=a(\rho)m(\theta),\quad u_\theta(\rho,\theta)=c(\rho)m'(\theta) \tag{11}$$

where the function $m(\theta)$ satisfies the Legendre equation [2,7,38]:

$$m''(\theta)+\cot\theta\cdot m'(\theta)+\left(z^2-\frac{1}{4}\right)m(\theta)=0. \tag{12}$$

Substituting (11) into (3) and (10), taking into account (12), the following boundary value problems with the spectral parameter, z, are obtained:

$$\frac{d}{d\rho}\left[b_{11}\frac{da}{d\rho}+\varepsilon\ b_{12}\left(2a-\left(z^2-\frac{1}{4}\right)c\right)\right]+\varepsilon\left[(b_{11}-2b_{12})\frac{da}{d\rho}+2(b_{12}-b_{22}-b_{23})\varepsilon\ a\right]-\left(z^2-\frac{1}{4}\right)\varepsilon\left[\varepsilon(b_{12}-b_{22}-b_{23}-b_{44})c+\varepsilon\ b_{44}a+b_{44}\frac{dc}{d\rho}\right]=0$$
$$\frac{d}{d\rho}\left[b_{44}\left(\frac{dc}{d\rho}+\varepsilon(a-c)\right)\right]+\varepsilon\left[b_{12}\frac{da}{d\rho}+\varepsilon(b_{22}+b_{23}+2b_{44})a++2b_{44}\frac{dc}{d\rho}+\varepsilon(b_{22}-b_{23}-2b_{44})c\right]-\left(z^2-\frac{1}{4}\right)\varepsilon^2b_{22}c=0, \tag{13}$$

$$\left[b_{11}\frac{da}{d\rho} + \varepsilon\ b_{12}\left(2a - \left(z^2 - \frac{1}{4}\right)c\right)\right]\Bigg|_{\rho=\pm 1} = 0\,,$$
$$b_{44}\left[\frac{dc}{d\rho} + \varepsilon(a - c)\right]\Bigg|_{\rho=\pm 1} = 0\,. \tag{14}$$

The piecewise continuity of the elastic characteristics gives rise to difficulties in the investigation of Eqs. (13) and (14). To solve the Eqs. (13) and (14), we use the asymptotic method based on three iterative processes [6,26,27,31].

The HSs, which are appropriate to the first iterative process, can be obtained from (6)–(8) if we put $f^{\pm}(\theta) = t^{\pm}(\theta) = 0$ in them, we have

$$u_\rho^{(1)} = B\left[\cos\theta\ \ln\left(\cot\frac{\theta}{2}\right) - 1\right],$$
$$u_\theta^{(1)} = B\left[\sin\theta\ \ln\left(\cot\frac{\theta}{2}\right) + \cot\theta\right]. \tag{15}$$

These solutions correspond to the eigenvalues $z = 1.5$. The stresses corresponding to these solutions have the form:

$$\sigma_{\rho\rho}^{(1)} = \sigma_{\rho\theta}^{(1)} = 0\,,\quad \sigma_{\theta\theta}^{*(1)} = \frac{(b_{22} - b_{23})e^{-\varepsilon\rho}}{\sin^2\theta}B,\quad \sigma_{\varphi\varphi}^{*(1)} = \frac{(b_{23} - b_{22})e^{-\varepsilon\rho}}{\sin^2\theta}B\,. \tag{16}$$

Let us turn to the construction of the second SSS. The solution of Eqs. (13) and (14) is sought as:

$$a^{(2)}(\rho) = a_{20}(\rho) + \varepsilon\ a_{21}(\rho) + \ldots\,,\qquad c^{(2)}(\rho) = \varepsilon\left[c_{20}(\rho) + \varepsilon\ c_{21}(\rho) + \ldots\right], \tag{17}$$

$$z = \varepsilon^{-\frac{1}{2}}(\alpha_0 + \varepsilon\alpha_1 + \ldots)\,. \tag{18}$$

Substituting (17) and (18) into (13) and (14), after some transformations we obtain:

$$u_\rho^{(2)}(\rho,\theta) = \sum_{j=1}^{4} T_j a_j^{(2)}(\rho) m_j(\theta),\quad u_\theta^{(2)}(\rho,\theta) = \sum_{j=1}^{4} T_j c_j^{(2)}(\rho) m_j'(\theta) \tag{19}$$

where

$$a_j^{(2)}(\rho) = 1 + \varepsilon\left[\frac{\alpha_{0j}^2 p_1 + b_{22}^{(0)} - b_{23}^{(0)}}{p_0}\int_0^\rho \frac{b_{12}}{b_{11}}dx - \alpha_{0j}^2\int_0^\rho \frac{b_{12}}{b_{11}}xdx\right] + O(\varepsilon^2),$$
$$c_j^{(2)}(\rho) = \varepsilon\left[\frac{\alpha_{0j}^2 p_1 - g_0}{\alpha_{0j}^2 p_0} - \rho + O(\varepsilon)\right]$$

To determine α_{0j}, we obtain the following biquadratic equation:

$$[p_0 p_2 + b_{22}^{(1)}(b_{23}^{(1)} - g_1)]\ \alpha_{0j}^4 + (g_1 p_0 - g_0 p_1)\ \alpha_{0j}^2 + 2g_0 p_0 - g_0^2 = 0. \tag{20}$$

The stresses corresponding to solutions (19) take the form:

$$\sigma_{\rho\rho}^{(2)} = \varepsilon\sum_{j=1}^{4} T_j\left\{\frac{(-g_0 + \alpha_{0j}^2 p_1)}{p_0}\left[-\alpha_{0j}^2\int_{-1}^{\rho}\left(\int_{-1}^{y}\frac{(b_{12}^2 - b_{11}b_{22})}{b_{11}}dx\right)dy\right.\right.$$
$$\left. - \int_{-1}^{\rho}\frac{(b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2)}{b_{11}}dx\right]$$
$$+ \alpha_{0j}^4\int_{-1}^{\rho}\left(\int_{-1}^{y}\frac{(b_{12}^2 - b_{11}b_{22})}{b_{11}}xdx\right)dy$$
$$+ \alpha_{0j}^2\left[-\int_{-1}^{\rho}\left(\int_{-1}^{y}\frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}dx\right)dy\right.$$
$$\left. + \int_{-1}^{\rho}\frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}xdx\right] + 2\int_{-1}^{\rho}\frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}dx$$
$$\left. + O(\varepsilon)\right\}m_j(\theta),$$

$$\sigma_{\rho\theta}^{(2)} = \varepsilon\sum_{j=1}^{4} T_j\left\{-\int_{-1}^{\rho}\frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}dx + \alpha_{0j}^2\int_{-1}^{\rho}\frac{b_{12}^2 - b_{11}b_{22}}{b_{11}}xdx\right.$$
$$\left. - \frac{(-g_0 + \alpha_{0j}^2 p_1)}{p_0}\int_{-1}^{\rho}\frac{b_{12}^2 - b_{11}b_{22}}{b_{11}}dx + O(\varepsilon)\right\}m_j'(\theta),$$

$$\sigma_{\varphi\varphi}^{*(2)} = \sum_{j=1}^{4} T_j\left\{\left[\frac{(-g_0 + \alpha_{0j}^2 p_1)}{p_0}\frac{(b_{12}^2 - b_{11}b_{23})}{b_{11}} - \alpha_{0j}^2\frac{(b_{12}^2 - b_{11}b_{23})}{b_{11}}\rho\right.\right.$$
$$+ \frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}$$
$$\left. + O(\varepsilon)\right]m_j(\theta) + \varepsilon\left[(b_{22} - b_{23})\left(\frac{-g_0 + \alpha_{0j}^2 p_1}{\alpha_{0j}^2 p_0} - \rho\right)\cot\theta + O(\varepsilon)\right]m_j'(\theta)\right\},$$

$$\sigma_{\theta\theta}^{*(2)} = \sum_{j=1}^{4} T_j\left\{\left[\frac{-(g_0 + \alpha_{0j}^2 p_1)}{p_0}\cdot\frac{(b_{12}^2 - b_{11}b_{22})}{b_{11}} + \frac{b_{11}b_{22} + b_{11}b_{23} - 2b_{12}^2}{b_{11}}\right.\right.$$
$$- \alpha_{0j}^2\rho$$
$$\left.\times\frac{(b_{12}^2 - b_{11}b_{22})}{b_{11}} + O(\varepsilon)\right]m_j(\theta)$$
$$\left. + \varepsilon\left[(b_{23} - b_{22})\left(\frac{-g_0 + \alpha_{0j}^2 p_1}{\alpha_{0j}^2 p_0} - \rho\right)\cot\theta + O(\varepsilon)\right]m_j'(\theta)\right\}. \tag{21}$$

It is noted that the asymptotic expansion of $q(\theta,\lambda)\sim\sum_{k=0}^{\infty} q_k(\theta)\lambda^k$ is valid, as $q_0(\theta) \neq 0(\theta_1 < \theta < \theta_2)$ and $\mathrm{Re}(\lambda^{-1}\sqrt{q_0(\theta)}) \geq 0(\lambda \in (0;\lambda_0))$ and $\lambda \to 0$, then the principal term of the asymptotic solution of equation

$$\lambda^2 y''(\theta) - q(\theta,\lambda)y(\theta) = 0 \tag{22}$$

at $\lambda \to 0$ has the form [39]:

$$y_{1,2}(\theta,\lambda) = q_0^{-0.25}(\theta)\exp\left[\pm\lambda^{-1}\int_{\theta_0}^{\theta}\sqrt{q_0(t)}dt\right.$$
$$\left. + \frac{1}{2}\int_{\theta_0}^{\theta}\frac{q_1(t)}{\sqrt{q_0(t)}}dt\right][1 + O(\varepsilon)]. \tag{23}$$

Substitution

$$m(\theta) = \frac{y(\theta)}{\sqrt{\sin\theta}} \tag{24}$$

leads Eq. (12) to the form:

$$y''(\theta) + \left(z^2 + \frac{1}{4\sin^2\theta}\right)y(\theta) = 0 \tag{25}$$

Substituting (18) into (25), using (23) and taking into account (24), we obtain that for the second iterative process the principal term of the asymptotic solution of the Eq. (12) with $\varepsilon \to 0$, takes the form:

$$m_k(\theta) = \begin{cases} \frac{1}{\sqrt{\sin\theta}}\frac{1}{\sqrt[4]{-\alpha_{0k}^2}}\exp\left[-\varepsilon^{-\frac{1}{2}}\sqrt{-\alpha_{0k}^2}(\theta-\theta_1)\right]\left(1+O\left(\varepsilon^{\frac{1}{2}}\right)\right) \\ in \text{ neighborhood } \theta=\theta_1, \\ \frac{1}{\sqrt{\sin\theta}}\frac{1}{\sqrt[4]{-\alpha_{0k}^2}}\exp\left[\varepsilon^{-\frac{1}{2}}\sqrt{-\alpha_{0k}^2}(\theta-\theta_2)\right]\left(1+O\left(\varepsilon^{\frac{1}{2}}\right)\right) \\ in \text{ neighborhood } \theta=\theta_2. \end{cases} \tag{26}$$

According to the third iterative process, the solution (13) and (14) is sought as:

$$a^{(3)}(\rho) = a_{30}(\rho) + \varepsilon a_{31}(\rho) + \dots, \quad c^{(3)}(\rho) = \frac{\varepsilon}{\beta_0}[c_{30}(\rho) + \varepsilon c_{31}(\rho) + \dots], \tag{27}$$

$$z = i\varepsilon^{-1}(\beta_0 + \varepsilon\beta_1 + \dots). \tag{28}$$

After substituting (27) and (28) into Eqs. (13) and (14) for the first terms, we obtain:

$$N(\beta_0)\overline{w_0} = \{h_1(\beta_0)\bar{w}_0, \quad h_2(\beta_0)\bar{w}_0 = \bar{0} \quad at \quad \rho = \pm 1\} = \bar{0} \tag{29}$$

where

$$h_1(\beta_0)\overline{w_0} = (B_0 + \beta_0 B_1 + \beta_0^2 B_2)\,\overline{w_0}, \quad h_2(\beta_0)\overline{w_0} = (C_0 + \beta_0 C_1)\overline{w_0}$$

in which

$$B_0 = \left\|\begin{matrix}\partial(b_{11}\partial) & 0 \\ 0 & \partial(b_{44}\partial)\end{matrix}\right\|,$$

$$B_1 = \left\|\begin{matrix}0 & b_{44}\partial + \partial(b_{12}) \\ \partial(b_{44}) + b_{12}\partial & 0\end{matrix}\right\|,$$

$$B_2 = \left\|\begin{matrix}b_{44} & 0 \\ 0 & b_{22}\end{matrix}\right\|,$$

$$C_0 = \left\|\begin{matrix}b_{11}\partial & 0 \\ 0 & b_{44}\partial\end{matrix}\right\|, \quad C_1 = \left\|\begin{matrix}0 & b_{12} \\ b_{44} & 0\end{matrix}\right\|, \quad \overline{w_0} = (a_{30}, \ c_{30})^T$$

The spectral problem (29), describes a potential solution of a transversally-isotropic plate inhomogeneous in the thickness direction [15,17,19,20,26,27,31,34–37]. By replacing

$$a_{30}(\rho) = -\beta_0^{-3}[e_0\psi''(\rho)]' + \beta_0^{-1}b_{44}^{-1}\psi'(\rho) + \beta_0^{-1}[e_1\psi(\rho)]',$$
$$c_{30}(\rho) = \beta_0^{-2}e_0\psi''(\rho) - e_1\psi(\rho) \tag{30}$$

The spectral problem (29) reduces to the following problem:

$$[e_0\psi''(\rho)]'' - \beta_0^2\{[e_1\psi(\rho)]'' + e_1\psi''(\rho) + [b_{44}^{-1}\psi'(\rho)]'\} + \beta_0^4 e_2\psi(\rho) = 0$$
$$\psi'(\rho)|_{\rho=\pm1} = 0, \quad \beta_0\psi(\rho)|_{\rho=\pm1} = 0. \tag{31}$$

where

$$e_0 = \frac{b_{11}}{b_{12}^2 - b_{11}b_{22}}, \quad e_1 = \frac{b_{12}}{b_{12}^2 - b_{11}b_{22}}, \quad e_2 = \frac{b_{22}}{b_{12}^2 - b_{11}b_{22}}.$$

The Eq. (31) is a generalization of the spectral problem of Papkovich for the inhomogeneous transversely-isotropic materials [15,31,34–37]. In the next step we obtain the boundary value problem for determining $\bar{w}_1 = (a_{31}, \ c_{31})^T$ and β_1:

$$(B_0 + \beta_0 B_1 + \beta_0^2 B_2)\overline{w_1} = [\rho(B_0 + \beta_0 B_1 + \beta_0^2 B_2) - 2\beta_0\beta_1 B_2 + A_0 + \beta_0 A_1 + \beta_1 A_2]\overline{w_0}$$
$$(C_0 + \beta_0 C_1)\overline{w_1}|_{\rho=\pm1} = [\rho(C_0 + \beta_0 C_1) + \beta_1 C_3 + C_4]\overline{w_0}_{\rho=\pm1} \tag{32}$$

where

$$A_0 = \left\|\begin{matrix}-2\partial(b_{12}) + (2b_{12} - b_{11})\partial & 0 \\ 0 & \partial(b_{44}) - 2b_{44}\partial\end{matrix}\right\|$$

$$A_1 = \left\|\begin{matrix}0 & (b_{44} + b_{23} + b_{22} - b_{12}) \\ -(b_{22} + b_{23} + 2b_{44}) & 0\end{matrix}\right\|,$$

$$A_2 = \left\|\begin{matrix}0 & -2(\partial(b_{12}) + b_{44}\partial) \\ 0 & 0\end{matrix}\right\|,$$

$$C_3 = \left\|\begin{matrix}0 & -2b_{12} \\ 0 & 0\end{matrix}\right\|, \quad C_4 = \left\|\begin{matrix}-2b_{12} & 0 \\ 0 & b_{44}\end{matrix}\right\|$$

The condition for solvability of (32) is the orthogonality of the right-hand side of the solution of the dual problem:

$$N^*(\beta_0)\bar{w}_0^* = N(-\bar{\beta}_0)\bar{w}_0^* = \bar{0} \tag{33}$$

where

$\bar{w}_0^* = (a_{30}^*, \ c_{30}^*)^T$.

Satisfying this condition, for β_1 we have:

$$\beta_1 = \frac{N_2}{N_1} \tag{34}$$

where

$$N_1 = 2\int_{-1}^{1}\{(b_{12}c_{30})'\bar{a}_{30}^* + b_{44}c_{30}'\bar{a}_{30}^* + \beta_0(b_{44}a_{30}\bar{a}_{30}^* + b_{22}c_{30}\bar{c}_{30}^*)\}d\rho$$

$$N_2 = \int_{-1}^{1}\{\beta_0[(\rho b_{12}c_{30})'\bar{a}_{30}^* + \rho b_{44}c_{30}'\bar{a}_{30}^* + (\rho b_{44}a_{30})'\bar{c}_{30}^* + \rho b_{12}a_{30}'\bar{c}_{30}^* + (b_{44} + b_{23} + b_{22} - 2b_{12})c_{30}\bar{a}_{30}^* - (b_{22} + b_{23} + 3b_{44})a_{30}\bar{c}_{30}^* + \beta_0^2\rho(b_{44}a_{30}\bar{a}_{30}^* + b_{22}c_{30}\bar{c}_{30}^*) + (b_{44}c_{30})'\bar{c}_{30}^* + (\rho b_{11}a_{30}')'\bar{a}_{30}^* + (\rho b_{44}c_{30}')'\bar{c}_{30}^* - 3b_{44}c_{30}'\bar{c}_{30}^* - 2(b_{11} - b_{12})a_{30}'\bar{a}_{30}^* - 2(b_{12}a_{30})'\bar{a}_{30}^*\}d\rho$$

In the third iterative process, the solutions are:

$$u_\rho^{(3)}(\rho,\theta) = \sum_{k=1}^{\infty} D_k[-\beta_{0k}^{-3}(e_0\psi_k''(\rho))' + \beta_{0k}^{-1}b_{44}^{-1}\psi'_k(\rho) + \beta_{0k}^{-1}(e_1\psi_k(\rho))' + O(\varepsilon)] \ \times m_k(\theta)$$
$$u_\theta^{(3)}(\rho,\theta) = \varepsilon\sum_{k=1}^{\infty} D_k[\beta_{0k}^{-3}e_0\psi_k''(\rho) - \beta_{0k}^{-1}e_1\psi_k(\rho) + O(\varepsilon)] \ m_k'(\theta). \tag{35}$$

In the third iterative process, the stresses are expressed as:

$$\sigma_{\rho\rho}^{(3)} = \frac{1}{\varepsilon}\sum_{k=1}^{\infty} D_k[-\beta_{0k}\psi_k(\rho) + O(\varepsilon)]\,m_k(\theta),$$
$$\sigma_{\rho\theta}^{(3)} = \sum_{k=1}^{\infty} D_k[\beta_{0k}^{-1}\psi_k'(\rho) + O(\varepsilon)]\,m_k'(\theta),$$
$$\sigma_{\theta\theta}^{*(3)} = \frac{1}{\varepsilon}\sum_{k=1}^{\infty} D_k[-\beta_{0k}^{-1}\psi_k''(\rho) + O(\varepsilon)]\,m_k(\theta),$$
$$\sigma_{\varphi\varphi}^{*(3)} = \frac{1}{\varepsilon}\sum_{k=1}^{\infty} D_k\left[\frac{\beta_{0k}^{-1}(b_{11}b_{23} - b_{12}^2)}{b_{12}^2 - b_{11}b_{22}}\psi_k''(\rho) + \frac{\beta_{0k}\,b_{12}(b_{22} - b_{23})}{b_{12}^2 - b_{11}b_{22}}\psi_k(\rho) + O(\varepsilon)\right]m_k(\theta). \tag{36}$$

Substituting (28) into (25), using (23) and taking into account (24), we obtain that for the third iterative process the principal term of the asymptotic solution (12) with $\varepsilon \to 0$ takes the form:

$$m_k(\theta) = \begin{cases} \frac{1}{\sqrt{\beta_{0k}\sin\theta}}\exp[-\varepsilon^{-1}\sqrt{\beta_{0k}^2}(\theta-\theta_1)](1+O(\varepsilon)) \ in \text{ neighborhood } \theta = \theta_1. \\ \frac{1}{\sqrt{\beta_{0k}\sin\theta}}\exp[\varepsilon^{-1}\sqrt{\beta_{0k}^2}(\theta-\theta_2)](1+O(\varepsilon)) \ in \text{ neighborhood } \theta = \theta_2. \end{cases} \tag{37}$$

The general solution (13) and (14) is the sum of the solutions (15), (19) and (35) corresponding to the above three iterative processes [6]:

$$u_\rho(\rho,\theta)=\sum_{i=1}^{3}u_\rho^{(i)},\quad u_\theta(\rho,\theta)=\sum_{i=1}^{3}u_\theta^{(i)}$$

4. SSS corresponding to different types of HSs

We represent displacements as follows:

$$u_\rho=u_\rho^{(1)}+\sum\nolimits_{k=1}^{\infty}E_k a_k(\rho)\ m_k(\theta),\qquad u_\theta=u_\theta^{(1)}+\sum\nolimits_{k=1}^{\infty}E_k c_k(\rho)\ m'_k(\theta). \tag{38}$$

The second term includes the displacement determined by the second and third iterative process. For stresses we have:

$$\sigma_{\theta\theta}^{*}=\sigma_{\theta\theta}^{*(1)}+\sum_{k=1}^{\infty}E_k(\sigma_{1k}^{(1)}(\rho)m_k(\theta)+\sigma_{1k}^{(2)}(\rho)m_k'(\theta)\cot\theta),\quad \sigma_{\rho\theta}=\sum_{k=1}^{\infty}E_k\sigma_{2k}(\rho)m_k'(\theta) \tag{39}$$

where

$$\sigma_{1k}^{(1)}(\rho)=\frac{e^{-\varepsilon\rho}}{\varepsilon}\left[b_{12}a_k'(\rho)+\varepsilon(b_{22}+b_{23})a_k(\rho)-\varepsilon b_{22}\left(z_k^2-\frac{1}{4}\right)c_k(\rho)\right]$$
$$\sigma_{1k}^{(2)}(\rho)=e^{-\varepsilon\rho}(b_{23}-b_{22})\,c_k(\rho),$$
$$\sigma_{2k}(\rho)=\frac{e^{-\varepsilon\rho}}{\varepsilon}b_{44}\left[c_k'(\rho)+\varepsilon(a_k(\rho)-c_k(\rho))\right]$$

Let us consider the connection between HSs and the principal vector P of stresses in the section $\theta(=\text{const})$, is defined as

$$P=2\pi\sin\theta\int_{r_1}^{r_2}(\sigma_{r\theta}\cos\theta-\sigma_{\theta\theta}\sin\theta)rdr$$

or

$$P=2\pi\varepsilon\sin\theta\int_{-1}^{1}(\sigma_{\rho\theta}\cos\theta-\sigma_{\theta\theta}^{*}\sin\theta)\ e^{2\varepsilon\rho}d\rho. \tag{40}$$

Substituting (39) into (40), we obtain:

$$P=-4\pi\varepsilon B\int_{-1}^{1}G(\rho)\,e^{\varepsilon\rho}d\rho+2\pi\varepsilon\sin\theta\sum_{k=1}^{\infty}E_k[d_{2k}m_k'(\theta)\cos\theta-d_{1k}m_k(\theta)\sin\theta] \tag{41}$$

where

$$d_{1k}=\int_{-1}^{1}\sigma_{1k}^{(1)}(\rho)e^{2\varepsilon\rho}d\rho,\quad d_{2k}=\int_{-1}^{1}(\sigma_{2k}(\rho)-\sigma_{1k}^{(2)}(\rho))\,e^{2\varepsilon\rho}d\rho$$

Multiplying Eq. (13) by $e^{\varepsilon\rho}$ and integrating within [-1, 1], using integration by parts using the appropriate boundary conditions (14), we have:

$$2d_{1k}+\left(z^2-\frac{1}{4}\right)d_{2k}=0,\quad d_{1k}+d_{2k}=0 \tag{42}$$

It follows from (42) that

$$d_{1k}=d_{2k}=0 \tag{43}$$

On the basis of (41) and (43) for the principal vector, P, we obtain:

$$P=-4\pi\varepsilon B\int_{-1}^{1}G(\rho)\,e^{\varepsilon\rho}d\rho. \tag{44}$$

In section $\theta(=\text{const})$, the moment M and shearing force Q for solving the second and third iterative process have the form:

$$M^{(2)}=\varepsilon^2\sin\theta\sum\nolimits_{j=1}^{4}T_j\left[\frac{(p_1^2-p_0p_2)\alpha_{0j}^2+p_0g_1-p_1g_0}{p_0}\right]m_j(\theta)+O(\varepsilon^3)$$
$$Q^{(2)}=\varepsilon^2\sin\theta\sum\nolimits_{j=1}^{4}T_j\left[\frac{(p_1^2-p_0p_2)\alpha_{0j}^2+2p_0g_0-p_0g_1-g_0p_1}{p_0}\right]m_j'(\theta)+O(\varepsilon^3),$$
$$M^{(3)}=O(\varepsilon^3),\quad Q^{(3)}=O(\varepsilon^3) \tag{45}$$

From expression (45), we find that the principal parts of the moment and shear force determined by the solution of the second SSS. The solution of Eq. (15) corresponding to the first iterative process determines the internal SSS of the sphere. The stresses determined by the second and third SSSs are localized at the ends of the sphere (in conical sections $\theta=\theta_j(j=1,2)$). From (26) and (37), we find that when moving away from the conical sections $\theta=\theta_j(j=1,2)$, these solutions decrease exponentially, and the attenuation index of the stresses determined by the second SSS has the order of $O(\varepsilon^{-1/2})$ versus the ε, and the stresses corresponding to the third SSS are of order $O(\varepsilon^{-1})$. The stress state corresponding to the solutions of (19) is the edge effects in the applied shell theory [7,10,26,27,31]. The first terms of the expansion in ε of the solution (19) in combination with the first terms of (15) can be considered as solutions of the classical shell theory. The third asymptotic process determines the solutions (35) that have the character of the boundary layer, which in the Kirchhoff-Love theory are absent. The first terms of (36) are equivalent to the Saint-Venant edge effect of an inhomogeneous transversely-isotropic plate [31]. For imaginary β_{0k}, the boundary layer of Saint-Venant is weakly damped, and solutions (36) should be classified as internal solutions. In this case, the SSS of the transversely-isotropic and isotropic spheres is qualitatively different [31]. When β_{0k} are real or complex, the overall picture of the SSS are qualitatively similar to the corresponding picture for an isotropic sphere [1,2,7,10] and they differ in the rate of attenuation of the Saint-Venant boundary layers.

The above analysis shows that the stressed state of a RIHTIS consists of three types: a penetrating stress state, a simple edge effect, and a boundary layer.

5. Satisfaction of boundary conditions

We assume that the stresses of the sphere are given as

$$\sigma_{\theta\theta}^{*}|_{\theta=\theta_j}=f_{1j}(\rho),\qquad \sigma_{\rho\theta}|_{\theta=\theta_j}=f_{2j}(\rho). \tag{46}$$

where $f_{1j}(\rho)$ and $f_{2j}(\rho)$ $(j=1,\ 2)$ are sufficiently smooth functions having order ε relative to $O(1)$ and satisfying the following equilibrium conditions:

$$2\pi\sin\theta_1\int_{-1}^{1}(f_{21}(\rho)\cos\theta_1-f_{11}(\rho)\sin\theta_1)\,e^{2\varepsilon\rho}d\rho==2\pi\sin\theta_2\int_{-1}^{1}(f_{22}(\rho)\cos\theta_2-f_{12}(\rho)\sin\theta_2)\,e^{2\varepsilon\rho}d\rho \tag{47}$$

The connection between the constant B and the principal vector P is represented by the equation:

$$B=-\frac{P}{4\pi\varepsilon\int_{-1}^{1}G(\rho)\,e^{\varepsilon\rho}d\rho} \tag{48}$$

To determine the constants E_k in this study, we use the variational principle of Lagrange [1,2,7]:

$$\sum_{j=1}^{2}\int_{-1}^{1}[(\sigma_{\theta\theta}^{*}-f_{1j}(\rho))\delta u_\theta+(\sigma_{\rho\theta}-f_{2j}(\rho))\delta u_\rho]\,|_{\theta=\theta_j}e^{2\varepsilon\rho}d\rho=0. \tag{49}$$

Substituting (38) and (39) into (49) and counting δE_j as independent variations, we obtain from (49) an infinite system of linear algebraic equations:

$$\sum_{k=1}^{\infty} D_{jk} E_k = H_j, \quad (j = \overline{1, \infty}) \tag{50}$$

where

$$D_{jk} = \int_{-1}^{1} \sigma_{1k}^{(1)}(\rho)\, c_j(\rho)\, e^{2\varepsilon\rho} d\rho \cdot \sum_{s=1}^{2} m_k(\theta_s) m_j'(\theta_s) +$$
$$+ \int_{-1}^{1} \sigma_{1k}^{(2)}(\rho) c_j(\rho) e^{2\varepsilon\rho} d\rho \sum_{s=1}^{2} m_k'(\theta_s) m_j'(\theta_s) \cot\theta_s +$$
$$+ \int_{-1}^{1} \sigma_{2k}(\rho) a_j(\rho) e^{2\varepsilon\rho} d\rho \sum_{s=1}^{2} m_k'(\theta_s) m_j(\theta_s)$$
$$H_j = \sum_{s=1}^{2} \left[m_j'(\theta_s) \int_{-1}^{1} f_{1s}^{*}(\rho) c_j(\rho)\, e^{2\varepsilon\rho} d\rho + m_j(\theta_s) \int_{-1}^{1} f_{2s}(\rho) a_j(\rho)\, e^{2\varepsilon\rho} d\rho \right]$$

in which

$$f_{1s}^{*}(\rho) = f_{1s}(\rho) + \frac{PG(\rho)}{2\pi\varepsilon e^{\varepsilon\rho} \sin^2\theta_s \int_{-1}^{1} G(\rho) e^{\varepsilon\rho} d\rho}, \quad (s = 1, 2)$$

The system of infinite linear algebraic Eq. (50) is always solvable under physically meaningful conditions imposed on the right-hand side of Eq. (50). The solvability and convergence of the reduction method for Eq. (50) is proved in Refs. [12].

Using the smallness of the parameter, ε, we construct asymptotic $\sigma_{\rho\theta}^{(2)} = O(\varepsilon^{1/2})$ and $\sigma_{\theta\theta}^{*(2)} = O(1)$ solutions of the system of Eq. (50). Considering that, we will clarify the assumptions regarding the external load. The tangential stresses given at the ends are decomposed as:

$$f_{2s}(\rho) = f_{2s}^{(1)} + f_{2s}^{(2)}$$

where

$$f_{2s}^{(1)} = \int_{-1}^{1} f_{2s}(\rho) d\rho, \qquad f_{2s}^{(2)} = f_{2s}(\rho) - f_{2s}^{(1)}, \quad (s = 1, \ 2).$$

It can be shown that

$$\int_{-1}^{1} \sigma_{\rho\theta}\big|_{\theta=\theta_s} d\rho = \sum_{j=1}^{4} T_j \left(\int_{-1}^{1} \sigma_{2j}(\rho) d\rho \right) m_j'(\theta_s) = \frac{\sqrt{\varepsilon}\sqrt{-\alpha_{0j}^2}}{\sin\theta_s \sqrt[4]{-\alpha_{0j}^2}}$$
$$\times \sum_{j=1}^{4} T_j \left[g_0 - g_1 + \alpha_{0j}^2 (p_1 - p_2) + \frac{\alpha_{0j}^2 p_1 - g_0}{p_0}(p_1 - p_0) + O\left(\varepsilon^{\frac{1}{2}}\right) \right],$$
$$\left(s = 1, \ 2\right) \tag{51}$$

On the basis of the Eq. (51) we find that

$$f_{2s}^{(1)} = O(\varepsilon^{1/2}), \quad f_{2s}^{(2)} = O(1), \quad (s = 1, \ 2) \tag{52}$$

The unknown constants T_j and D_k $(j = \overline{1, 4}; \ k = 1, 2, \ldots)$ will be sought in the form:

$$T_j = T_{j0} + \varepsilon T_{j1} + \ldots, \tag{53}$$

$$D_k = \varepsilon D_{k0} + \varepsilon^2 D_{k1} + \ldots. \tag{54}$$

After substituting (53) and (54) into Eq. (50), taking into account (52) for the determination of T_{j0} and D_{k0}, we obtain the following systems of linear algebraic equations,

$$\sum_{j=1}^{4} l_{kj} T_{j0} = \tau_k' \quad (k = \overline{1; 4}), \tag{55}$$

$$\sum_{j=1}^{\infty} L_{kj} D_{k0} = \tau_j'' \quad (j = \overline{1; \infty}), \tag{56}$$

where

$$l_{kj} = \frac{\sqrt{-\alpha_{0j}^2} - \sqrt{-\alpha_{0k}^2}}{\sqrt[4]{\alpha_{0k}^2 \alpha_{0j}^2}} \quad \frac{\alpha_{0j}^2 (p_1^2 - p_0 p_2) + (p_1 g_0 - p_0 g_1)}{p_0},$$

$$\tau_k'$$
$$= \frac{\sin\theta_1 \sin\theta_2}{\sin\theta_1 - \sin\theta_2} \sum_{s=1}^{2} \frac{1}{\sqrt{\sin\theta_s}} \left[(-1)^s \frac{\sqrt{-\alpha_{0k}^2}}{\sqrt[4]{-\alpha_{0k}^2}} \right.$$
$$\left. \int_{-1}^{1} f_{1s}^{*}(\rho) \left(\frac{\alpha_{0k}^2 p_1 + g_0}{\alpha_{0k}^2 p_0} - \rho \right) d\rho + \frac{1}{\sqrt[4]{-\alpha_{0k}^2}} f_{2s}^{*} \right]$$

$$L_{kj} = \frac{1}{\sqrt{\beta_{0j}\beta_{0k}^3}} \int_{-1}^{1} \left[-e_1 \psi_k \psi_j'' + \frac{1}{b_{44}} \psi_j' \psi_k' + \beta_{0j}^2 e_2 \psi_k \psi_j \right] d\rho$$
$$+ \frac{1}{\sqrt{\beta_{0k}\beta_{0j}^3}} \int_{-1}^{1} \left[e_1 \psi_j'' \psi_k - \beta_{0j}^2 e_2 \psi_k \psi_j \right] d\rho$$

$$\tau_j'' = \frac{\sin\theta_1 \sin\theta_2}{(\sin\theta_1 - \sin\theta_2)\sqrt{\beta_{0j}}} \left\{ -\frac{1}{\sqrt{\sin\theta_1}} \int_{-1}^{1} f_{11}^{*}(\rho) \left[\beta_{0j}^{-2} e_0 \psi_j'' \right. \right.$$
$$-e_1 \psi_j] d\rho$$
$$+ \frac{1}{\sqrt{\sin\theta_2}} \int_{-1}^{1} f_{12}^{*}(\rho) \left[\beta_{0j}^{-2} e_0 \psi_j'' - e_1 \psi_j \right] d\rho$$
$$+ \frac{1}{\sqrt{\sin\theta_1}} \int_{-1}^{1} f_{21}(\rho) \left[-\beta_{0j}^{-3} \left(e_0 \psi_j'' \right)' \right.$$
$$\left. + \beta_{0j}^{-1} \frac{1}{b_{44}} \psi_j' + \beta_{0j}^{-1} (e_1 \psi_j)' \right] d\rho + \frac{1}{\sqrt{\sin\theta_2}} \int_{-1}^{1} f_{22}(\rho)$$
$$\left. \times \left[-\beta_{0j}^{-3} \left(e_0 \psi_j'' \right)' + \beta_{0j}^{-1} \frac{1}{b_{44}} \psi_j'' + \beta_{0j}^{-1} (e_1 \psi_j)' \right] d\rho \right\} \tag{57}$$

in which

$$f_{2s}^{(1)} = \varepsilon^{\frac{1}{2}} (f_{2s}^{*} + \ldots), \quad (s = 1, \ 2) \tag{58}$$

Definitions T_{jp} and D_{kp} $(p = 1, 2, \ldots)$ invariably reduces to systems whose matrices coincide with the matrices of the systems (55) and (56).

6. Results and discussion

6.1. Comparative studies

To verify the accuracy of this work, a comparison is made with the study of Mekhtiev [2], who using the homogeneous solutions method solved the problem of homogeneous transversely-isotropic spheres of small thickness. The asymptotic expressions (2.3.2), (2.3.3) and (2.3.5)-(2.3.7) are obtained for displacements and stresses of the homogeneous transversely-isotropic sphere of small thickness in the work [2] (See, Chapter 2 in the ref. [2]). If the $b_{ij}(\rho)$ are constant, from asymptotic formulas (15), (16), (19), (21), (35) and (36) for the displacements and stresses in the present work, we obtain corresponding expressions for the homogeneous transversely-isotropic spheres in the study [2], in the particular case. The symbols u_r, u_θ, σ_r, σ_θ, σ_φ, $\tau_{r\theta}$ for displacements and stresses are used in [2]. For numerical calculations are used the following data:

$$A_{11}^{(0)} = 5.13 \times 10^{11}, \ A_{22}^{(0)} = 12.1 \times 10^{11}, \ A_{23}^{(0)} = 4.81 \times 10^{11}, \ A_{12}^{(0)}$$
$$= 4.42 \times 10^{11}, \ A_{44}^{(0)} = 1.85 \times 10^{11}, \ \varepsilon = 0.2.$$

The above transversally-isotropic material properties are related to the cadmium material (Cd) and are taken from study [40]. As can be seen from the results presented in Table 1, these results are in very good agreement with the results obtained in the study of Mekhtiev [2]. That is, the stress values of a homogeneous transversely-isotropic sphere are very close in time to each other in both studies.

In addition, when the Young's moduli $b_{ij}(\rho)$ are constant and

Table 1
Comparision the values of stresses of H transversely-isotropic sphere with the results of Mekhtiev [2].

r	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
	Present study		
1.06	− 144.297	− 2.361	193.342
1.12	− 231.428	− 2.682	302.981
1.18	− 269.941	− 2.952	351.596
1.24	− 273.367	− 3.192	353.682
1.30	− 248.854	− 3.401	319.912
1.36	− 197.933	− 3.58	254.477
1.42	− 126.338	− 3.721	162.133
r	Study of Mekhtiev [2]		
1.06	− 144.298	− 2.363	193.344
1.12	− 231.427	− 2.681	302.982
1.18	− 269.943	− 2.954	351.596
1.24	− 273.366	− 3.193	353.683
1.30	− 248.852	− 3.404	319.914
1.36	− 197.934	− 3.582	254.476
1.42	− 126.337	− 3.723	162.134

$b_{11} = b_{22} = 2G + \lambda$, $b_{12} = b_{23} = \lambda$ and $b_{44} = G$ are taken into account, from the formulas for displacements and stresses of the present study, all results for homogeneous isotropic spheres of small thickness are obtained in the Ref. [7].

6.2. New computations and analysis for the stresses of RINHTIHSs

As an example, let us consider the problem of the SSS of radially inhomogeneous and homogeneous transversely-isotropic spheres with the small thickness. We study the following two cases:

Case 1. The region occupied by the sphere is $\Gamma = \{r \in [1,\ 1.5],\ \theta \in [3^\circ,\ 80^\circ],\ \varphi \in [0,\ 2\pi]\}$. The small parameter, ε, characterizing the thickness of the sphere is equal to 0.2, i.e., $\varepsilon = 0.2$.

Case 2. The region occupied by the sphere is $\Gamma = \{r \in [1,\ 1.04],\ \theta \in [3^\circ,\ 80^\circ],\ \varphi \in [0,\ 2\pi]\}$ and $\varepsilon = 0.02$.

We assume that the lateral surface of the sphere is stress free, and the boundary conditions are given on the conical sections as:

$$\sigma_{\theta\,\theta} = Kr^2 sin\left(\frac{\pi}{9}\right), \quad \sigma_{r\,\theta} = 0, \quad \text{when } \theta = 3^\circ$$

$$\sigma_{\theta\theta} = 0, \quad \sigma_{r\theta} = Kr^2\left[cos\left(\frac{\pi}{30}\right) - 1\right], \quad \text{when } \theta = 80^\circ$$

where K is the parameter and the value in the accounts is taken as equal to one.

We assume that for RINHTIHSs the Young's moduli change by quadratic laws along the radially coordinate, r [5]:

$$A_{11} = A_{11}^{(0)}\bar{r}^2, \quad A_{12} = A_{12}{}^{(0)}\bar{r}^2, \quad A_{22} = A_{22}{}^{(0)}\bar{r}^2, \quad A_{23} = A_{23}{}^{(0)}\bar{r}^2, \quad A_{44} = A_{44}{}^{(0)}\bar{r}^2.$$

where $A_{ij}^{(0)}(Pa)$, $(i, j = 1, 2, 3, 4)$ are the material properties of the homogeneous transversally-isotropic sphere. The Cadmium (Cd) is used as homogeneous transversally-isotropic material and properties defined as:

$$A_{11}^{(0)} = 5.13 \times 10^{11}, \quad A_{22}^{(0)} = 12.1 \times 10^{11}, \quad A_{23}^{(0)} = 4.81 \times 10^{11}, \quad A_{12}^{(0)} = 4.42 \times 10^{11}, \quad A_{44}^{(0)} = 1.85 \times 10^{11}.$$

The properties of Cadmium material (Cd) are taken from the study of Huntington [40]. In all the figures examined below, the values of stresses (Pa) are taken as $\sigma \times 10^8$.

The following expression is used in the calculation of the percentages: $\frac{\sigma^{INH} - \sigma^{H}}{\sigma^{H}} \times 100\%$.

Let us consider the Case 1. The stress distributions through the thickness of radially inhomogeneous (INH) and homogeneous (H)

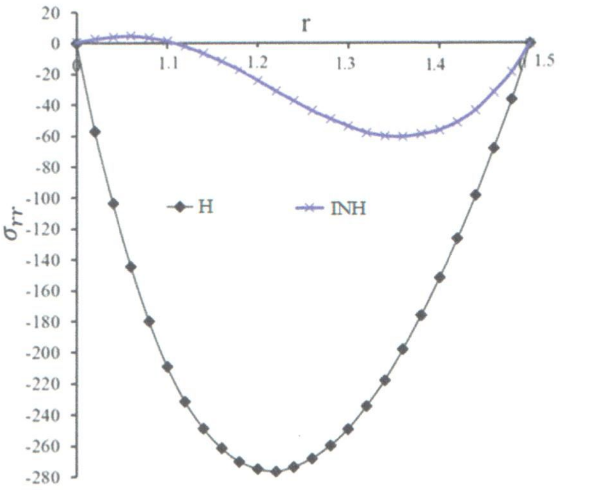

Fig. 2. Distribution of the σ_{rr} for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.2$.

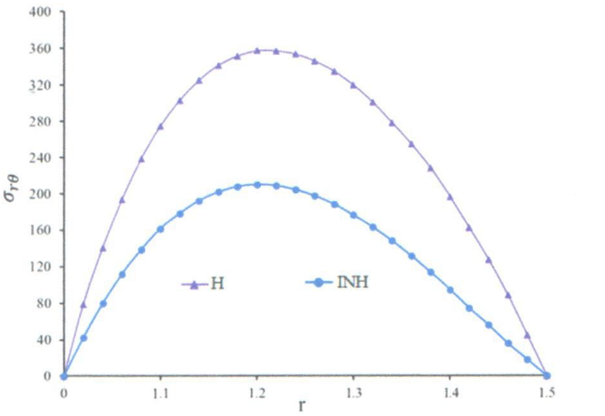

Fig. 3. Distribution of the $\sigma_{r\theta}$ for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.2$.

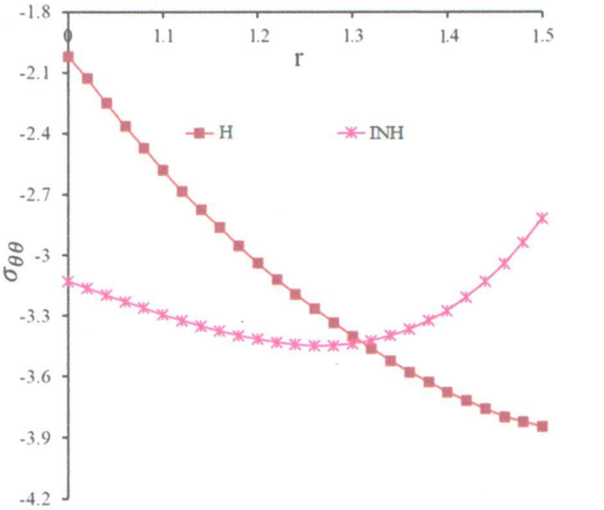

Fig. 4. Distribution of the $\sigma_{\theta\theta}$ for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.2$.

Table 2
Distribution of stresses of H and INH transversely-isotropic spheres versus r for $\varepsilon = 0.2$.

r	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
	Homogenous sphere		
1.0	0	0	− 2.018
1.1	− 208.767	274.726	− 2.578
1.2	− 274.613	357.445	− 3.039
1.3	− 248.854	319.912	− 3.401
1.4	− 151.679	196.092	− 3.679
1.5	0	0	− 3.849
r	Inhomogeneous sphere		
1.0	0	0	− 3.128
1.1	1.588	161.475	− 3.295
1.2	− 23.938	209.893	− 3.417
1.3	− 53.536	176.196	− 3.439
1.4	− 56.022	94.815	− 3.276
1.5	0	0	− 2.822

Table 3
Distribution of stresses of H and INH transversely-isotropic spheres versus r for $\varepsilon = 0.02$.

r	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
	Homogeneous sphere		
1.000	0	0	− 2.1884
1.005	− 0.545	3.531	− 2.1862
1.010	− 0.903	6.019	− 2.1835
1.015	− 1.092	7.456	− 2.1805
1.020	− 1.132	7.925	− 2.1769
1.025	− 1.029	7.376	− 2.1729
1.030	− 0.795	5.867	− 2.1686
1.035	− 0.445	3.375	− 2.1637
1.040	0	0	− 2.1588
r	Inhomogeneous sphere		
1.000	0	0	− 2.273
1.005	− 0.308	0.41	− 2.252
1.010	− 0.501	0.786	− 2.229
1.015	− 0.66	1.048	− 2.206
1.020	− 0.692	1.211	− 2.183
1.025	− 0.679	1.208	− 2.158
1.030	− 0.533	1.043	− 2.133
1.035	− 0.334	0.664	− 2.108
1.040	0	0	− 2.081

transversely-isotropic spheres depending on the radial coordinate are shown Figs. 2–4, respectively, at $\varepsilon = 0.2$.

In Figs. 2–4 and Table 2 show the distributions of stresses for H and INH transversally-isotropic spheres versus the radial coordinate, r. Figs. 2–4 are described stress distributions only along the thickness of the spheres. It can be seen from Fig. 2 that at a distance of 0.06 from the inner surface, the stress σ_{rr} for INH spheres assumes the largest value. The distribution of σ_{rr} for INH spheres at a distance of 0.16 from the inner surface varies almost in accordance with the quadratic law and at $r = 1.36$ it assumes the smallest value. The distribution of $\sigma_{r\theta}$ for INH spheres occurs according to a quadratic law and at $r = 1.2$ reaches its maximum value (Fig. 3). The distributions of $\sigma_{\theta\theta}$ for INH spheres to a distance of 0.06 from the outer surface vary almost quadratic law (Fig. 4). It can be seen from Fig. 2 that σ_{rr} for H spheres varies according to a law close to a square parabola and assumes the smallest value, at $r = 1.22$. The distribution of $\sigma_{r\theta}$ for H spheres occurs according to a quadratic law and reaches its maximum, at $r = 1.22$ (Fig. 3). The distribution of $\sigma_{\theta\theta}$ for H spheres occurs according to the law of inverse proportionality from distances (Fig. 4). It is seen from Fig. 4 that the stresses $\sigma_{\theta\theta}$ the inner spherical surface in the case of the homogeneous material are greater than the inhomogeneous material, and on the outer spherical surface it is smaller. It is seen from Table 2 also that the most pronounced effect of inhomogeneity is the value of the σ_{rr}, when compared to other stresses. For example, when the inhomogeneous

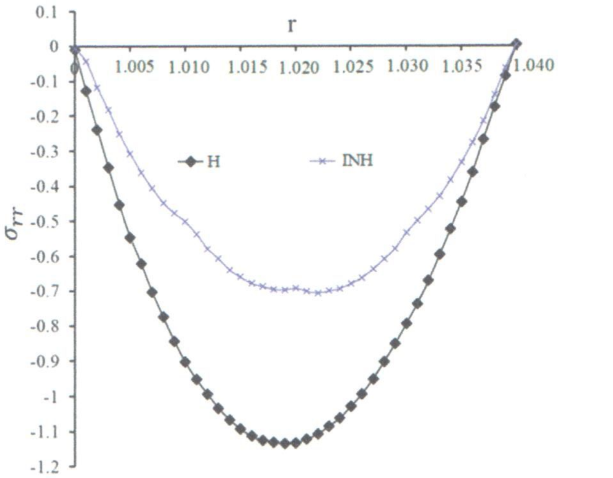

Fig. 5. Distribution of the σ_{rr} for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.02$.

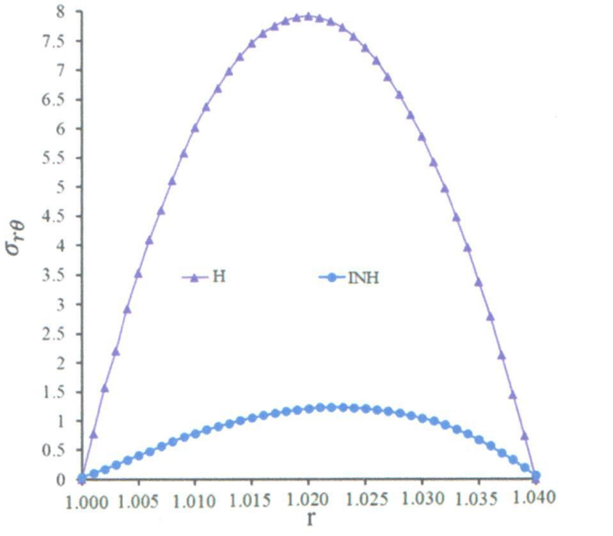

Fig. 6. Distribution of the $\sigma_{r\theta}$ for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.02$.

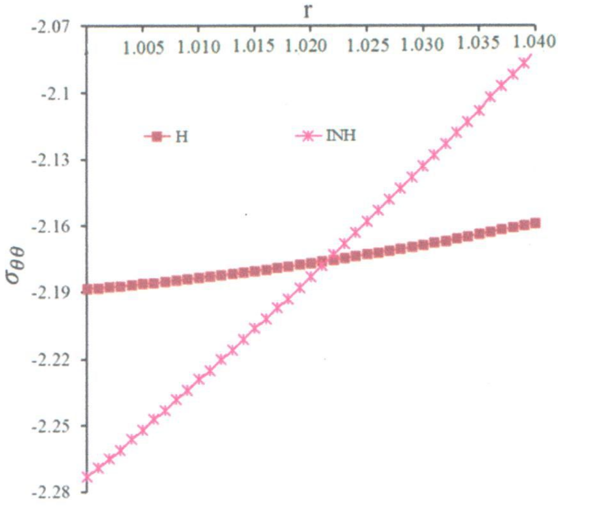

Fig. 7. Distribution of the $\sigma_{\theta\theta}$ for H and INH transversally-isotropic spheres versus the r with $\varepsilon = 0.02$.

transversely-isotropic spheres is compared with the homogeneous transversely-isotropic sphere, the greatest influence of the inhomogeneity; on the value of σ_{rr} is about (-100%) for $r = 1.1$, on the values of $\sigma_{r\theta}$ is about (+55%) for $r = 1.0$ and on the $\sigma_{\theta\theta}$ is about (-51.65%) for $r = 1.4$. When the values of $\sigma_{\theta\theta}$ for the inhomogeneous transversely-isotropic spheres are compared with the values of $\sigma_{\theta\theta}$ of the homogeneous transversely- isotropic sphere, the influence of the inhomogeneity on the minimum and maximum values of $\sigma_{\theta\theta}$ are about 55.01% and 26.68%, respectively.

Let us consider Case 2. The stress distributions for H and INH transversally-isotropic spheres, at $\varepsilon = 0.02$ are presented in Figs. 5–7 and Table 3. From Fig. 5, we see that the distributions of σ_{rr} for H and INH transversally-isotropic spheres occur according to the quadratic law and the branches of the parabola are directed upward. At $r = 1.023$, the stress σ_{rr} takes the lowest value. The distribution of $\sigma_{r\theta}$ for H and INH transversally-isotropic spheres occur according to the quadratic law and the branches of the parabola are directed downward. At the $r = 1.02$, the stress $\sigma_{r\theta}$ takes the greatest value (Fig. 6). The distributions of the stress, $\sigma_{\theta\theta}$ for H and INH transversally-isotropic spheres at the distance of 0.02 from the inner surface vary almost linearly (Fig. 7). In the case of $\varepsilon = 0.02$, the distributions of the stresses $\sigma_{\theta\theta}$ and σ_{rr} for H and INH transversally-isotropic spheres are qualitatively different (Table 3).

7. Conclusions

In this study, the three-dimensional problem of the theory of elasticity for radially inhomogeneous transversally-isotropic thin hollow spheres is investigated using the asymptotic integration method. First, the basic relations and equilibrium equations for radially inhomogeneous transversally-isotropic thin hollow spheres are formed. After obtaining inhomogeneous and HSs, the nature of the constructed HSs is studied. On the basis of the theoretical analysis, it is shown that the SSS in a radially inhomogeneous transversely-isotropic hollow sphere consists of three types: a penetrating stress state, a simple edge effect, and a boundary layer. From the analysis of numerical results it follows that the inhomogeneity of the material can have a significant effect on the SSS.

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